# Making and breaking post-quantum cryptography from elliptic curves

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# Recall: Diffie–Hellman key exchange '76

Public parameters:

- a finite group $G$ (typically $\mathbb{F}_q^*$ or $E(\mathbb{F}_q)$)
- an element $g \in G$ of (large) prime order $p$

| Alice | public | Bob |
|---|---|---|
| $a \xleftarrow{\text{random}} \{0 \ldots p-1\}$ | | $b \xleftarrow{\text{random}} \{0 \ldots p-1\}$ |
| $g^a$ | $g^a \rightarrow$, $\leftarrow g^b$ | $g^b$ |
| $s := (g^b)^a$ | | $s := (g^a)^b$ |

The Discrete Logarithm Problem, finding $a$ given $g$ and $g^a$, should be hard[1] in $\langle g \rangle$.

[1] Complexity (at least) subexponential in $\log(p)$.

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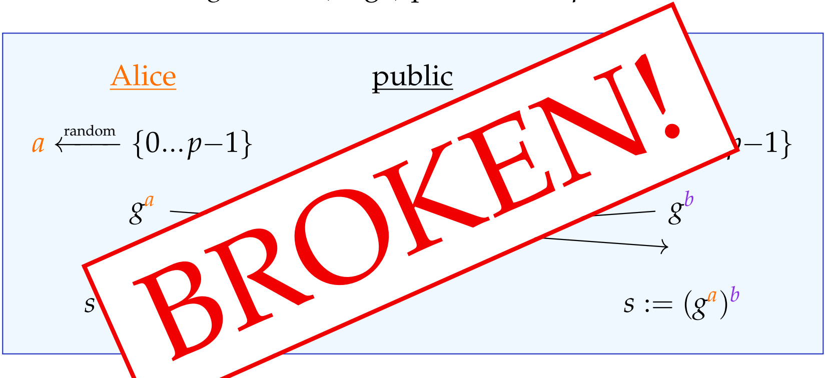


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# Quantumifying Exponentiation

- Couveignes ‘97, Rostovtsev, Stolbunov ‘04: Idea to replace the Discrete Logarithm Problem: replace exponentiation

$$\begin{array}{rcl} \mathbb{Z} \times G & \to & G \\ (x, g) & \mapsto & g^x \end{array}$$

by a group action on a set.

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  by a group action on a set.
- Replace $G$ by a set $S$ of specially chosen elliptic curves $/\mathbb{F}_q$.
- Replace $\mathbb{Z}$ by a commutative group $H$ that acts freely and transitively on $S$ via surjective morphisms (isogenies):

$$\begin{array}{rcl} H \times S & \to & S \\ (\alpha, E) & \mapsto & \alpha * E := \alpha(E) \end{array}$$

# Couveignes-Rostovstev-Stolbunov key exchange

Public parameters:

- a finite set $S$ (of specially chosen elliptic curves $/\mathbb{F}_q$),
- an element $E \in S$,
- a group $H$ that acts freely and transitively on $S$ via $*$.

| Alice | public | Bob |
| --- | --- | --- |
| $\alpha \xleftarrow{\text{random}} H$ | | $\beta \xleftarrow{\text{random}} H$ |
| $\alpha * E$ | $\alpha * E \longrightarrow$ Bob; $\beta * E \longrightarrow$ Alice | $\beta * E$ |
| $s := \alpha * (\beta * E)$ | | $s := \beta * (\alpha * E)$ |

Finding $\alpha$ given $E$ and $\alpha * E$, should be hard.[2]

[2]Complexity (at least) subexponential in $\log(\#S)$.

# From CRS to CSIDH

1997 Couveignes proposes the now-CRS scheme.

- Uses ordinary elliptic curves/$\mathbb{F}_p$ with same end ring.
- Paper is rejected and forgotten.

2004 Rostovstev, Stolbunov rediscover now-CRS scheme.

- Best known quantum and classical attacks are exponential.

2005 Kuperberg: quantum subexponential attack for the dihedral hidden subgroup problem.

2010 Childs, Jao, Soukharev apply Kuperberg to CRS.

- Secure parameters $\rightsquigarrow$ key exchange of 20 minutes.

2011 Jao, De Feo propose SIDH [more to come!].

2017 De Feo, Kieffer, Smith use modular curves to do a CRS key exchange in 8 minutes.

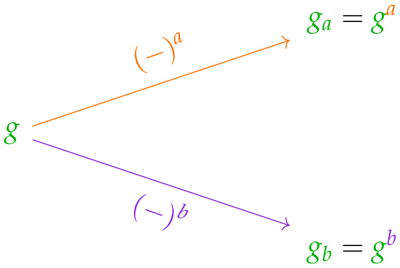
2018 Castryck, Lange, M., Panny, Renes propose CSIDH.

- CRS but with supersingular elliptic curves $/\mathbb{F}_p$.
- $p$ constructed to make scheme efficient.
- Key exchange runs in 60ms.*


[ˈsiːˌsaɪd]

# Evolution of key exchange

**Diffie-Hellman**

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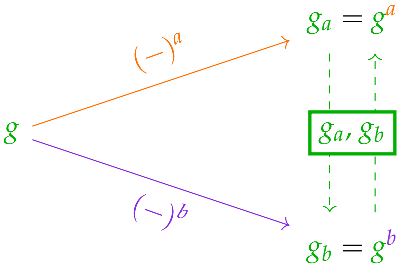


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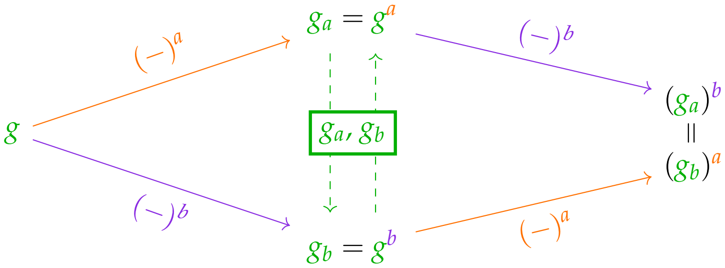


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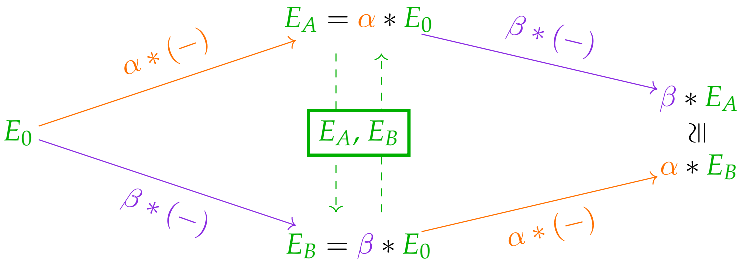
## CRS or CSIDH

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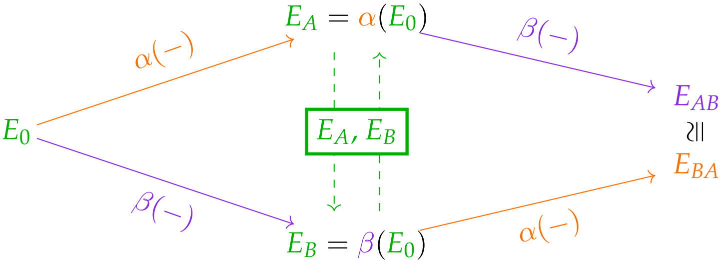
# Evolution of key exchange

## From CRS to SIDH

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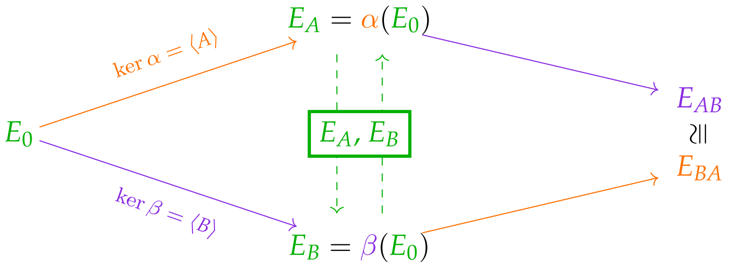


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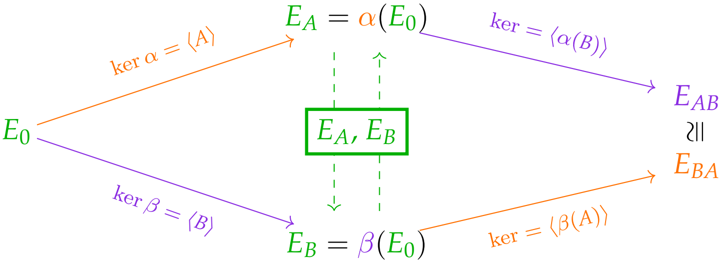


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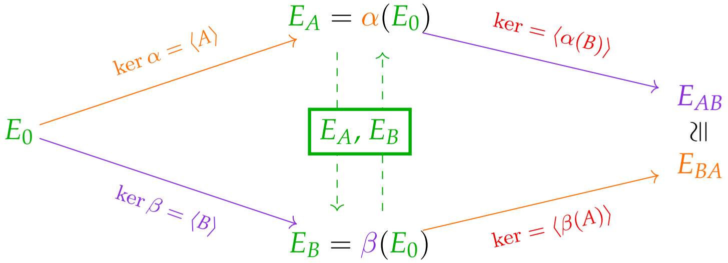


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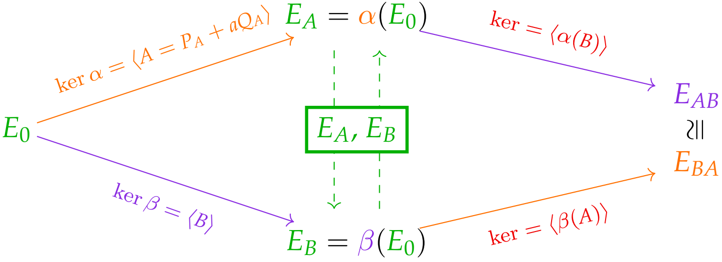


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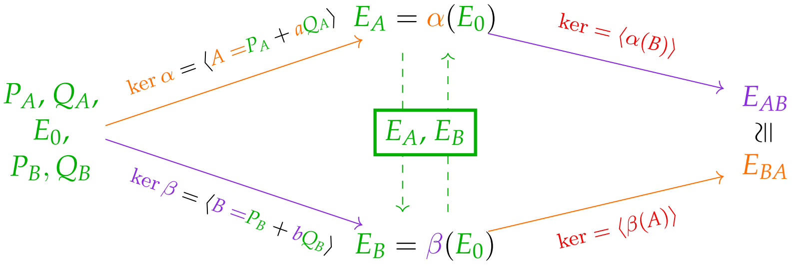


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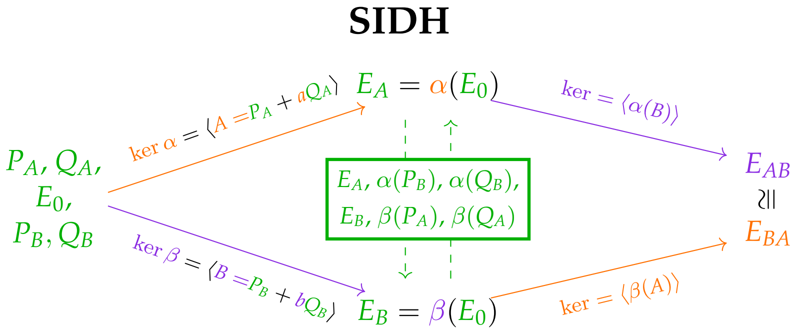


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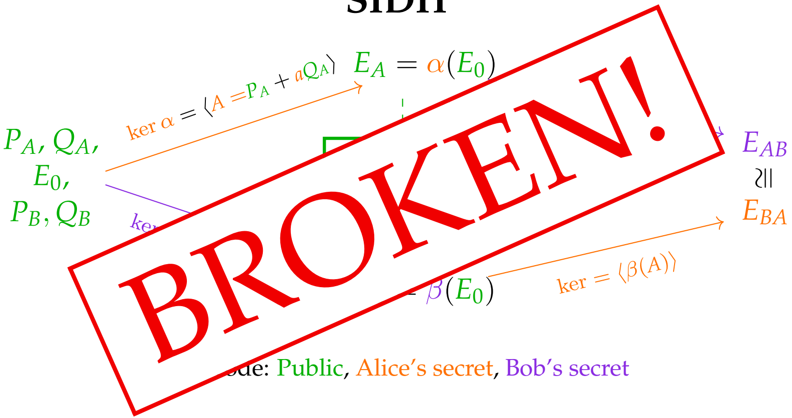

# Summary of hard problems

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- SIDH –

There are public elliptic curves $E_0$ and $E_A$, and a secret isogeny $\alpha : E_0 \to E_A$. Given the points $P_B$, $Q_B$ on $E_0$ and $\alpha(P_B)$, $\alpha(Q_B)$, compute $\alpha$. (modulo technical restrictions)*

*Details for the elliptic curve lovers:

$p$ a large prime; $E_0/\mathbb{F}_{p^2}$ and $E_A/\mathbb{F}_{p^2}$ supersingular; $\deg(\alpha)$, $N$ public large smooth coprime integers; points $P_B$, $Q_B$ chosen such that $\langle P_B, Q_B \rangle = E_0[N]$.

# History of the SIDH problem

2011 Problem introduced by De Feo, Jao, and Plut

2016 Galbraith, Petit, Shani, Ti give active attack

2017 Petit gives passive attack on some parameter sets

2020 de Quehen, Kutas, Leonardi, M., Panny, Petit, Stange give passive attack on more parameter sets

2022 Castryck-Decru and Maino-M. give passive attack on SIKE parameter sets; Robert extends to all parameter sets

- CD and MM attack is subexponential in most cases
- CD attack polynomial-time when $\mathrm{End}(E_0)$ known
- Robert attack polynomial-time in all cases
- Panny and Pope implement MM attack; Wesolowski independently discovers direct recovery method

# Technical interlude

There are public elliptic curves $E_0$ and $E_A$, and a secret isogeny $\alpha : E_0 \to E_A$. Given the points $P_B$, $Q_B$ on $E_0$ and $\alpha(P_B)$, $\alpha(Q_B)$, compute $\alpha$ . (modulo technical restrictions)*

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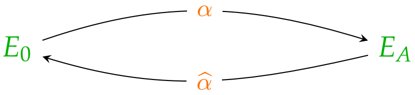
$\rightsquigarrow$ Petit's idea: Construct $\theta : E_A \to E_A$ such that $\ker(\widehat{\alpha}) \subseteq \ker(\theta)$.

# Petit's trick: torsion points to isogenies

Finding the secret isogeny $\alpha$ of known degree.



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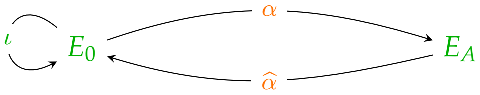


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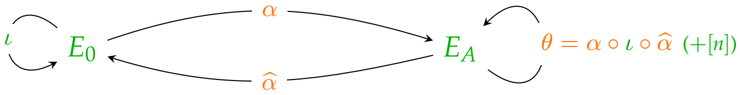


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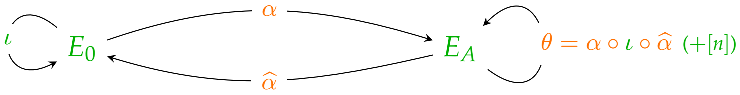


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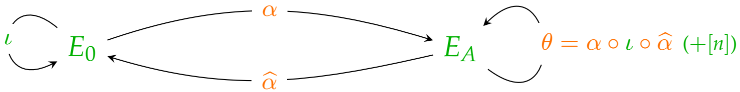


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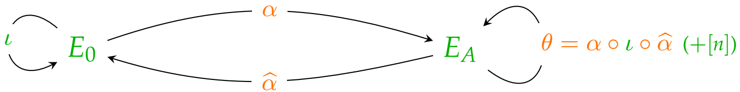


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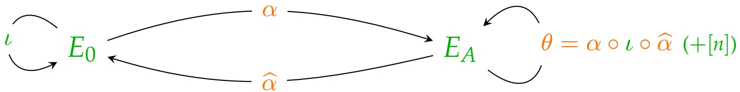


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- Restriction # 2 rules out SIKE parameters, where $N \approx \deg(\alpha)$ (and $p \approx N \cdot \deg \alpha$).

# Enter Kani

There are public elliptic curves $E_0$ and $E_A$, and a secret isogeny $\alpha : E_0 \to E_A$. Given the points $P_B$, $Q_B$ on $E_0$ and $\alpha(P_B)$, $\alpha(Q_B)$, compute $\alpha$. (modulo technical restrictions)*

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Solution? $\theta : E_0 \times E_A \to E_0 \times E_A$?
$\rightsquigarrow$ still not enough. But! Kani's theorem:

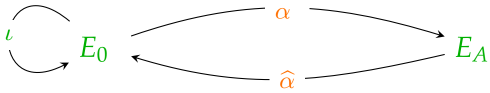
- Constructs $E_1$, $E_2$ such that there exists a (structure-preserving) isogeny

$$E_1 \times E_A \to E_0 \times E_2$$

  of the right degree, $N^2$.
- Petit's trick then applies.

# Recovering the secret

Finding the secret isogeny $\alpha$ of known degree.



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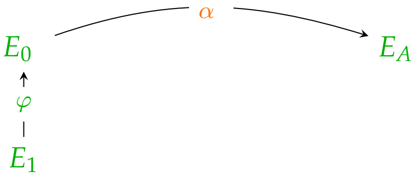

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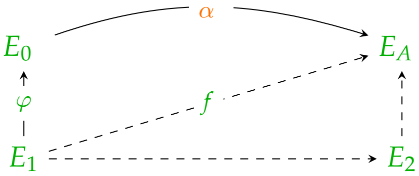


Kani's theorem constructs the above such that

$$\Phi = \begin{pmatrix} \varphi & -\widehat{\alpha} \\ * & * \end{pmatrix} : E_1 \times E_A \to E_0 \times E_2$$

is a structure preserving isogeny of degree $N^2$, and

$$\ker(\Phi) = \{(\deg(\alpha)P, f(P)) : P \in E_1[N]\}$$

$\rightsquigarrow$ can compute $\Phi$ and read off secret $\alpha$!

# Recovering the secret with Robert's trick

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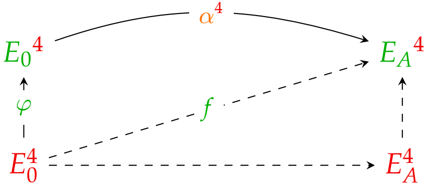


constructs the above such that

$$\Phi = \begin{pmatrix} \varphi & -\widehat{\alpha}^4 \\ * & * \end{pmatrix} : E_0^4 \times E_A^4 \to E_0^4 \times E_A^4$$

is a structure preserving isogeny of degree $N^2$, and

$$\ker(\Phi) \text{ is known}$$

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# What next?

- Fouotsa, Moriya, and Petit proposed mitigations
  - Masks either torsion point images or isogeny degrees
  - The mitigations make SIKE/SIDH unusably slow and big
  - For advanced protocols may still be a good option (c.f. Basso's OPRF, threshold schemes, etc.)
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  - Cryptanalysis ongoing effort
- Constructive applications?
  - FESTA: New PKE. Fast and small as SIKE was?
  - SQISignHD: Small, fast signatures with clean security reduction.
  - VDF-like construction.
  - Work in progress with Maino and Robert $\rightsquigarrow$ computing genus 2 cyclic isogenies.

# What next?

- Fouotsa, Moriya, and Petit proposed mitigations
  - Masks either torsion point images or isogeny degrees
  - The mitigations make SIKE/SIDH unusably slow and big
  - For advanced protocols may still be a good option (c.f. Basso's OPRF, threshold schemes, etc.)
  - Cryptanalysis ongoing effort
- Constructive applications?
  - FESTA: New PKE. Fast and small as SIKE was?
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  - VDF-like construction.
  - Work in progress with Maino and Robert $\rightsquigarrow$ computing genus 2 cyclic isogenies.

Thank you!