# Rings in which every ideal has two generators

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**Bachelor project**

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June 5, 2014

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# 1. Introduction

During an algebra course about modules, one learns that every finitely generated module over a principal ideal domain has a decomposition as a direct sum of its torsion submodule and a free module. In particular, if the module is also torsion-free, then it is a free module, hence a product of copies of the ring itself. It is then natural to ask whether we can generalize this structure theorem to domains that are not necessarily principal ideal domains.

Both Bass and Matlis have shown that rings in which every ideal can be generated by two elements, under varying assumptions on the rings, have the property that every finitely generated torsion-free module is isomorphic to a direct sum of ideals. Bass has shown this in the case of commutative rings $R$ with no nilpotent elements such that the integral closure is a finitely generated $R$-module, while Matlis has shown this in the case of local domains.

Moreover, Matlis has shown that these two properties, that every ideal can be generated by two elements and that every finitely generated torsion-free module is isomorphic to a direct sum of ideals, are equivalent for local domains. Bass has shown that they are equivalent under his conditions, except in a situation that can be analyzed completely. Additionally, they have both added a third equivalent property, but we will not mention the third equivalent property of Bass, as it is beyond the scope of this Bachelor thesis.

We shall mainly focus on the proof of Matlis, and more specifically, we will prove only one implication under some simplifying conditions in *Section 5*. Before we start on the proof however, we will first give some examples of domains in which every ideal can be generated by two elements in *Section 3*. We will then describe the dual and double dual of finitely generated modules in *Section 4*. Moreover, in *Section 6* we will consider the generalisation of the main theorem of this thesis, as stated by Bass, and we shall give the first part of the proof.

Before we state the main theorem however, we first need some definitions. We assume that the reader knows the basics of rings, integral domains and modules, including the definitions of a maximal ideal, a local ring, a semi-local ring and finitely generated modules. If this is not the case, then we refer to the appendices in *Section A* and *Section B*. All the other definitions required for the main theorem, including the main theorem itself, will be stated in *Section 2*.

## 2. Definitions & the main theorem

**Definition 2.1.** *Let $R$ be a commutative ring and let $n \in \mathbf{Z}_{\geqslant 0}$. A **chain of prime ideals of length** $n$ is a collection $\{\mathfrak{p}_i\}_{i=0}^n$ of prime ideals with the property that $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \ldots \subsetneq \mathfrak{p}_n$. We define the **Krull-dimension of** $R$ as the supremum of the lengths of all the chains of prime ideals in $R$.*

**Definition 2.2.** *Let $R$ be an integral domain and let $M$ be an $R$-module. A **torsion element** of $M$ is an $m \in M$, such that there exists an $r \in R \setminus \{0\}$ with $rm = 0$. If $0 \in M$ is the only torsion element, then $M$ is **torsion-free**.*

**Definition 2.3.** *Let $R$ be a commutative ring and $A$ an $R$-module. We define the **dual of** $A$ **with respect to** $R$ as $A^* := \operatorname{Hom}_R(A, R)$. If the canonical map*

$$\varphi : A \to A^{**} := (A^*)^*,$$
$$m \mapsto (f \mapsto f(m)),$$

*is injective, then $A$ is said to be **torsionless**, and if $\varphi$ is an isomorphism, then $A$ is said to be **reflexive**.*

**Remark 2.4.** *If $R$ is a domain, then it is straightforward to check that every torsion-less $R$-module is torsion-free. The converse holds if we add the condition that the module is finitely generated. For this, see Lemma 4.7.*

**Definition 2.5.** *Let $S$ be a multiplicatively closed subset of a ring $R$ with $1 \in S$. The **localisation of** $R$ **with respect to** $S$ is the $R$-algebra*

$$S^{-1}R := \{\langle r, s\rangle : r \in R, s \in S\} / \sim,$$

*where $\sim$ is the equivalence relation*

$$\langle r, s\rangle \sim \langle r', s'\rangle \quad \Leftrightarrow \quad \exists t \in S : t(rs' - r's) = 0,$$

*and the ring homomorphism is given by*

$$\pi : R \to S^{-1}R,$$
$$r \mapsto \langle r, 1\rangle .$$

*Addition and multiplication on $S^{-1}R$ are defined by*

$$\langle r, s\rangle + \langle r', s'\rangle := \langle rs' + sr', ss'\rangle, \quad \text{and} \quad \langle r, s\rangle \cdot \langle r', s'\rangle := \langle rr', ss'\rangle .$$

**Remark 2.6.** *If $R$ is an integral domain and $S$ is a multiplicatively closed subset of $R$ with $1 \in S$ and $0 \notin S$, then $\langle r, s\rangle \sim \langle r', s'\rangle$ if and only if $rs' = r's$. In particular, in that case, the canonical ring homomorphism $\pi$ is injective, and we then identify $R$ as a subring of $S^{-1}R$.*

**Notation 2.7.** *We write $r/s$ for the equivalence class of $\langle r, s\rangle$. Moreover, for every prime ideal $\mathfrak{p} \subsetneq R$, the set $S := R \backslash \mathfrak{p}$ is multiplicatively closed with $1 \in S$ and $0 \notin S$, and we define $R_{\mathfrak{p}} := S^{-1}R$.*

**Example 2.8.** *If $R$ is an integral domain, then $(0)$ is a prime ideal of $R$. We then define the **field of fractions** of $R$ as $K := R_{(0)}$.*

**Example 2.9.** *Let $R$ be an integral domain and $S \subseteq R$ a multiplicative closed subset of $R$. Denote the canonical ring homomorphism $R \to S^{-1}R$ by $\pi$. As can be seen in* [2, Proposition 2.2b]*, for all prime ideals $\mathfrak{p}$, the assignment $\mathfrak{q} \mapsto \pi^{-1}(\mathfrak{q})$ with $\mathfrak{q}$ a prime ideal of $R_{\mathfrak{p}}$, restricts to a one-to-one correspondence between prime ideals of $R_{\mathfrak{p}}$ and prime ideals of $R$ contained in $\mathfrak{p}$. The inverse of this assignment maps a prime ideal $\mathfrak{q}$ of $R$ containing $\mathfrak{p}$ to $\langle(\pi(q_i))_{q_i \in \mathfrak{q}}\rangle$. It follows that $R_{\mathfrak{p}}$ is a local ring with maximal ideal $\mathfrak{p}R_{\mathfrak{p}}$.*

**Definition 2.10.** *Let $R$ be an integral domain with field of fractions $K$ and let $A$ be an $R$-module. We define the* ***rank*** *of $A$ as $\operatorname{rank}(A) := \dim_K(K \otimes A)$.*

**Example 2.11.** *Non-zero ideals of an integral domain have rank $1$. Moreover, if $R$ is an integral domain and $K$ its field of fractions, then $R^n \otimes K = K^n$, hence $R^n$ has rank $n$. Finally note that finitely generated modules over integral domains have finite rank by Lemma B.2 of the appendix.*

**Definition 2.12.** *An integral domain is called* ***reflexive*** *if every torsionless $R$-module of finite rank is reflexive.*

**Definition 2.13.** *An integral domain $R$ is said to have* ***property FD*** *if every finitely generated torsion-free $R$-module is a direct sum of ideals. Furthermore, we say that $R$ has* ***property FD locally*** *if $R_M$ has property FD for every maximal ideal $M$ of $R$.*

**Definition 2.14.** *An integral domain is called* ***Noetherian*** *if every ideal is finitely generated.*

**Definition 2.15.** *Let $B$ be a ring with subring $R$. We define the* ***integral closure of*** *$R$* ***in*** *$B$ as the set of elements $b \in B$ such that there exists a monic polynomial $f$ over $R$ with $f(b) = 0$.*

We are now able to state the main theorem, which was given by Matlis [3]. Matlis himself was inspired by a similar theorem of Bass in [1], and the theorem is both more and less general than the one given by Bass. Bass has proven that a commutative ring $R$ with no nilpotent elements such that every ideal can be generated by two elements, has property FD, while the theorem of Matlis states this only in the case of local domains. The theorem of Matlis is however also more general, as it does not require that the integral closure of $R$ is a finitely generated $R$-module.

**Theorem 2.16** (Matlis, [3])**.** *Let $R$ be an integral domain with field of fractions $K$. Then the following are equivalent:*
*(1) Every ideal of $R$ can be generated by two elements.*
*(2) Every ring extension $S$ of $R$ in $K$ that is finitely generated as an $R$-module is reflexive and satisfies $\bigcap_{n \geqslant 0} I^n = 0$ for every $R$-ideal $I \subsetneq S$.*
*(3) The ring $R$ has property FD locally and is Noetherian of Krull dimension $1$.*

We shall prove that if $R$ is a local Noetherian integral domain satisfying (2), then $R$ has property FD. For this, see *Theorem 5.1.* The following definitions are important for the proof.

**Definition 2.17.** *An $R$-module $M$ is said to be* ***indecomposable*** *if $M \neq 0$ and $M$ cannot be written as a direct sum of two non-zero $R$-submodules.*

**Definition 2.18.** *Let $R$ be an integral domain with field of fractions $K$ and let $I$ and $J$ be $R$-submodules of $K$. We define $IJ$ to be the $R$-module generated by the elements $i \cdot j$ with $i \in I$ and $j \in J$. We say that $I$ is a* ***fractional ideal*** *if there exists a non-zero $r \in R$ such that $rI \subseteq R$. In particular, if both $I$ and $J$ are fractional ideals, then so is $IJ$. Moreover, if $I$ is a fractional ideal, then we define the $R$-module $I^{-1} := \{x \in K : xI \subseteq R\}$, which is a fractional ideal if $I \neq 0$. A fractional $R$-ideal $I$ is said to be* ***invertible*** *if there exists an $R$-module $J$ contained in $K$ such that $IJ = R$.*

**Remark 2.19.** *Let $R$ be an integral domain with fractional $R$-ideal $I$. If $I$ is invertible with $IJ = R$ for an $R$-module $J$, then $J = I^{-1}$. Indeed, we have $II^{-1} \subseteq R$, and multiplying both sides by $J$ yields $I^{-1} \subseteq J$. The inclusion $J \subseteq I^{-1}$ follows by definition of $I^{-1}$, hence $J = I^{-1}$. In particular, this justifies calling $I^{-1}$ the inverse of $I$ if $I$ is invertible.*

## 3. Using the theorem for concrete examples

3.1. **Rings of rank 2.** In this section we will consider the case where $R$ is an algebra over a principal ideal domain (PID) $P$ such that $R$ is free of rank 2 over $P$.

**Lemma 3.1.** *Let $P$ be a PID and let $R$ be a $P$-algebra. If $R \cong_P P^2$, then every ideal $I \subseteq R$ is of the form $I = x_1 R + x_2 R$ with $x_1, x_2 \in R$.*

*Proof.* Every $P$-submodule of $P^2$ is free of rank $k \leqslant 2$ by a direct generalization of [5, Theorem 9.7], which states that any subgroup of $\mathbf{Z}^n$ is free of rank $k \leqslant n$. As $R$-ideals are $P$-submodules of $R$ by definition, there exist $x_1, x_2 \in R$ such that $I = x_1 P + x_2 P$. Since $P$ is contained in $R$, we have $I \subseteq x_1 R + x_2 R$. Finally note that $x_1$ and $x_2$ are contained in $I$, hence $x_1 R + x_2 R \subseteq I$. Combining the two inclusions yields $I = x_1 R + x_2 R$. □

The following lemma gives us insight into rings of rank 2.

**Lemma 3.2.** *Let $P$ be a PID and let $R$ be a $P$-algebra. Then we have $R \cong P^2$ as $P$-modules if and only if $R \cong P[X]/(X^2 + bX + c)$ as rings with $b, c \in P$.*

*Proof.* If $R \cong_P P^2$, then there exist $x_1, x_2 \in R$ such that $R = x_1 P \oplus x_2 P$. We claim that, without loss of generalisation, we can assume $x_1 = 1$. To this end, note that there exists $a, b \in P$ such that $1 = x_1 a + x_2 b$, and we claim that $a$ and $b$ are coprime. If they are not coprime, then we define $1 \neq g := \text{ggd}(a, b)$. It follows that $g$ is a product of at least 1 irreducible element, hence $g$ is not a unit in $R$. As we assumed that $a$ and $b$ are not coprime, we can divide by $g$, hence $g^{-1} = g^{-1} x_1 a + g^{-1} x_2 b \in R$. Since $g$ is not a unit in $R$, we get a contradiction, so we conclude that $a$ and $b$ are coprime. In particular, there exist $c, d \in P$ such that $ac + bd = 1$. Let these $c$ and $d$ be given, we then define the basis transformation matrix

$$A := \begin{pmatrix} a & b \\ -d & c \end{pmatrix}.$$

We have $\det A = 1$ and

$$A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a x_1 + b x_2 \\ -d x_1 + c x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ c x_2 - d x_1 \end{pmatrix},$$

so we can indeed assume without loss of generalisation that $x_1$ equals 1. It follows that there exist $q, r \in P$ such that $x_2^2 = q x_2 + r$. Let these $q, r$ be given and define

$$\varphi : P[X] \to R,$$
$$x \mapsto x_2.$$

It is clear that $\varphi$ is a surjective ring homomorphism and by construction we have $\ker \varphi = (X^2 - qX - r)$. By the isomorphism theorem we conclude that $R$ is isomorphic to $P[X]/(X^2 - qX - r)$ as rings, so the first implication follows. The other implication of the lemma is straightforward to check. □

**Remark 3.3.** *Let $P$ be a PID and let $R$ be a $P$-algebra. If $R$ is isomorphic to $P^2$ as $P$-modules, then every ideal can be generated by two elements by Lemma 3.1. Moreover, by Lemma 3.2 we conclude that $R$ is isomorphic to $P[X]/(f)$ as rings for a monic polynomial $f \in P[X]$ of degree 2. It follows that $R$ is a domain if and only if $f$ is irreducible.*

A concrete example of a ring $R$ such that $R \cong P^2$ as $P$-modules with $P$ a PID is the ring $\mathbf{Z}[\sqrt{-3}]$. Let $\rho$ be a root of $x^2+x+1$ with $2\rho = -1+\sqrt{-3}$. We have $\sqrt{-3} = 2\rho+1$, hence $\mathbf{Z}[\sqrt{-3}] = \mathbf{Z} + 2\rho\mathbf{Z}$. We conclude that $\mathbf{Z}[\sqrt{-3}]$ is a subring of $\mathbf{Z}[\rho]$ of index 2.

**Definition 3.4.** *We define the* ***absolute value*** $|\cdot| : \mathbf{Q}(\sqrt{-3}) \to \mathbf{Q}$ *by*

$$|a+b\sqrt{-3}| = \sqrt{a^2+3b^2}.$$

*Note that this definition coincides with the absolute value on* $\mathbf{C}$.

Before we prove what every ideal of $\mathbf{Z}[\sqrt{-3}]$ looks like, we first prove a technical lemma. The idea behind the proof of this lemma is the same as in the proof that $\mathbf{Z}[i]$ is a PID, which is proven in [4, Theorem 12.19].

**Lemma 3.5.** *For every $z \in \mathbf{Q}(\sqrt{-3})$, there exists a $q \in \mathbf{Z}[\sqrt{-3}]$ such that either $|z-q| < 1$ or $z-q = \rho$.*

*Proof.* For any $z = a+b\sqrt{-3} \in \mathbf{Q}(\sqrt{-3})$, with $a, b \in \mathbf{Q}$, there exist $a_0, b_0 \in \mathbf{Z}$ such that $|a-a_0| \leqslant \frac{1}{2}$ and $|b-b_0| \leqslant \frac{1}{2}$. It follows that $x = a_0 + b_0\sqrt{-3}$ is an element of $\mathbf{Z}[\sqrt{-3}]$ such that

$$|z-x|^2 = |(a-a_0)+(b-b_0)\sqrt{-3}|^2 = |a-a_0|^2 + 3|b-b_0|^2 \leqslant \left(\tfrac{1}{2}\right)^2 + 3\left(\tfrac{1}{2}\right)^2 = 1,$$

where equality holds if and only if $a, b \in \frac{1}{2} + \mathbf{Z}$, that is, if and only if there exists a $q \in \mathbf{Z}[\sqrt{-3}]$ such that $z - q = \rho$, and the claim follows. □

**Lemma 3.6.** *Every ideal $I \subseteq \mathbf{Z}[\sqrt{-3}]$ is of the form $x\mathbf{Z}[\sqrt{-3}]$ or $x\mathbf{Z}[\rho]$ for an $x \in \mathbf{Z}[\sqrt{-3}]$.*

*Proof.* Let $I \subseteq \mathbf{Z}[\sqrt{-3}]$ be an arbitrary ideal. If $I = 0$, then certainly $I$ can be written as $0 \cdot \mathbf{Z}[\sqrt{-3}]$ and $0 \cdot \mathbf{Z}[\rho]$, so assume $I \neq 0$. Let $x \in I$ be non-zero such that $x$ has the smallest absolute value. We claim that $I = x\mathbf{Z}[\sqrt{-3}]$ or $I = x\mathbf{Z}[\rho]$ holds. To this end, we will first prove $I \subseteq x\mathbf{Z}[\rho]$. Let $y \in I$ and define $z = y/x$. By *Lemma 3.5*, there exists a $q \in \mathbf{Z}[\sqrt{-3}]$ such that either $|z-q| < 1$ or $z-q = \rho$. In the first case, we multiply both sides by $|x|$ to get

$$|y-qx| = |zx-qx| = |x||z-q| < |x|.$$

Note that $y-qx$ is an element of $I$ by construction, hence $y = qx$ by the minimality of $x$, and it follows that $y \in x\mathbf{Z}[\rho]$. In the latter case, we multiply both sides by $x$ to get $y-qx = xz-qx = \rho x$, hence $y = (q+\rho)x$, so $y \in x\mathbf{Z}[\rho]$. Since $x\mathbf{Z}[\sqrt{-3}] \subseteq I$ holds by definition of an ideal, we conclude $x\mathbf{Z}[\sqrt{-3}] \subseteq I \subseteq x\mathbf{Z}[\rho]$. As we have already shown $[\mathbf{Z}[\rho] : \mathbf{Z}[\sqrt{-3}]] = 2$, we conclude that every ideal $I$ is of the form $x\mathbf{Z}[\sqrt{-3}]$ or $x\mathbf{Z}[\rho]$ for an $x \in \mathbf{Z}[\sqrt{-3}]$. □

**Corollary 3.7.** *Every finitely generated torsion-free module over $\mathbf{Z}[\sqrt{-3}]$ is of the form $\mathbf{Z}[\sqrt{-3}]^{r_1} \oplus \mathbf{Z}[\rho]^{r_2}$ with $r_1, r_2 \in \mathbf{Z}_{\geqslant 0}$.*

*Proof.* We will show that $\mathbf{Z}[\sqrt{-3}]$ satisfies all the conditions of *Theorem 6.1*, the global version of *Theorem 2.16*, proven by Bass in [1, Section 7]. First of all, we claim that $\mathbf{Z}[\sqrt{-3}]$ has Krull dimension 1. To this end, note that $\mathbf{Z}[\sqrt{-3}]/(x)$ is finite if $x \in \mathbf{Z}[\sqrt{-3}] \setminus \{0\}$. As every non-zero prime ideal contains a non-zero principal ideal, it follows that $\mathbf{Z}[\sqrt{-3}]/\mathfrak{p}$ is a finite integral domain, hence a field, for every non-zero prime ideal $\mathfrak{p}$. We conclude that all prime ideals of $\mathbf{Z}[\sqrt{-3}]$ are maximal, thus $\mathbf{Z}[\sqrt{-3}]$ has Krull dimension 1.

Note that we have the inclusions $\mathbf{Z}[\sqrt{-3}] \subseteq \mathbf{Z}[\rho] \subseteq \mathbf{Q}[\sqrt{-3}]$ of rings. As $\mathbf{Z}[\rho]$ is a PID by [4, Exercise 12.58c], it follows that $\mathbf{Z}[\rho]$ is integrally closed, hence the integral closure of $\mathbf{Z}[\sqrt{-3}]$ inside $\mathbf{Q}[\sqrt{-3}]$ is $\mathbf{Z}[\rho]$. Since $\mathbf{Z}[\sqrt{-3}]$ is a subring of index 2 of $\mathbf{Z}[\rho]$, we conclude that $\mathbf{Z}[\rho]$ is finitely generated as a $\mathbf{Z}[\sqrt{-3}]$-module. Finally, as $\mathbf{Z}[\sqrt{-3}]$ is Noetherian by *Lemma 3.6*, we can use *Theorem 6.1.* Since $x\mathbf{Z}[\sqrt{-3}] \cong \mathbf{Z}[\sqrt{-3}]$ and $x\mathbf{Z}[\rho] \cong \mathbf{Z}[\rho]$ as $\mathbf{Z}[\sqrt{-3}]$-modules for all non-zero $x \in \mathbf{Z}[\sqrt{-3}]$, the claim follows. □

3.2. **Rings of power series without linear coefficient.** In this section we shall prove that every ideal of the ring of power series without linear coefficients over a field $K$ can be generated by two elements. To this end, we first need two important results.

**Lemma 3.8.** *Let $K$ be a field, let $R = K[[X]]$ and consider $f = \sum_{n \geqslant 0} a_n x^n \in R$ with $a_i \in K$. If $a_0 \neq 0$, then $f \in R^*$.*

*Sketch of the proof.* The proof of this lemma is straightforward. One recursively constructs a power series $g = \sum_{n \geqslant 0} b_n x^n$ starting with $b_0 = a_0^{-1}$ such that $fg = 1$. □

Using this lemma one can prove the following theorem.

**Theorem 3.9.** *If $K$ is a field, then $R = K[[X]]$ is a principal ideal domain and every ideal $I \subseteq R$ is of the form $I = (x^k)$ for a $k \in \mathbf{Z}_{\geqslant 0}$.*

*Sketch of the proof.* Pick an $f = \sum_{i \geqslant n} a_n x^n \in I$, with $I$ an ideal of $R$, such that $n$ is minimal. Multiplying by $a_n^{-1}$ then yields $a_n^{-1} f = x^n (1 + \sum_{i \geqslant n+1} \frac{a_i}{a_n} x^{i-n})$, where the latter term is an element of $R^*$ by *Lemma 3.8.* □

We now have enough tools for the following theorem.

**Theorem 3.10.** *Let $K$ be a field and let $R = \{a_0 + a_1 X + \ldots \in K[[X]] : a_1 = 0\}$. Every ideal of $R$ can be generated by two elements.*

*Proof.* By *Theorem 3.9* it follows that $P = K[[X^2]]$ is a P.I.D. As $R = P \oplus X^3 P$, we conclude that $R$ is isomorphic to $P^2$ as $P$-modules. The claim then follows by *Lemma 3.1.* □

3.3. **A non-example.** Let $S$ be a ring such that $S \cong \mathbf{Z}^n$ as additive abelian groups with $n \in \mathbf{N}$. Let $m \in \mathbf{N}$ and define $I := mS$ and $R := \mathbf{Z} + I$. It follows that $I$ is an $R$-ideal, and we claim that $I$ cannot be generated by $n-1$ or fewer elements as an $R$-ideal if $m > 1$.

*Proof.* We prove the claim by contradiction, so let $m > 1$ and assume $I$ can be generated by $n-1$ elements as an $R$-ideal. Since $I$ and $S$ are $R$-modules such that $I \subseteq S$, we can consider the quotient $R$-module $S/I$, which is isomorphic to $mS/mI = I/m^2S$ as $R$-modules. As $I$ can be generated by $n-1$ elements as an $R$-module, we conclude that $I/m^2S$ can also be generated by $n-1$ elements, so in particular $S/I$ too as they are isomorphic. Since $R = \mathbf{Z} + I$, it follows that $S/I$ can be generated by $n-1$ elements as a $\mathbf{Z}$-module. As $\mathbf{Z}$-modules however, we have $S/I \cong \mathbf{Z}^n/m\mathbf{Z}^n = (\mathbf{Z}/m\mathbf{Z})^n$ and as $(\mathbf{Z}/m\mathbf{Z})^n$ cannot be generated by less than $n$ elements, we get a contradiction. □

## 4. Describing the double dual of a module

In this section we will characterize the double dual $A^{**} := (A^*)^*$ of an $R$-module $A$, which will come in handy for our proof. To this end, we first need the definition of the localization of a module.

**Definition 4.1.** *Let $S$ be a multiplicatively closed subset of a commutative ring $R$ with $1 \in S$ and let $A$ be an $R$-module. The* ***localisation of** $A$ **with respect to** $S$* *is the $S^{-1}R$-module*

$$S^{-1}A := \{\langle a, s\rangle : a \in A, s \in S\} / \sim,$$

*where $\sim$ is the equivalence relation*

$$\langle a, s\rangle \sim \langle a', s'\rangle \quad \Leftrightarrow \quad \exists t \in S : t(s'a - sa') = 0.$$

*Addition and multiplication by $S^{-1}R$ are defined by*

$$\langle a, s\rangle + \langle a', s'\rangle := \langle s'a + sa', ss'\rangle\,, \quad \textit{and} \quad \langle r, s\rangle \cdot \langle a, s'\rangle := \langle ra, ss'\rangle\,.$$

**Notation 4.2.** *We write $a/s$ for the equivalence class of $\langle a, s\rangle$. Moreover, for every prime ideal $\mathfrak{p} \subset R$, the set $S := R \setminus \mathfrak{p}$ is multiplicatively closed with $1 \in S$, and we define $A_{\mathfrak{p}} := S^{-1}A$. By* [2, Proposition 2.4]*, it follows that $A_{\mathfrak{p}}$ is canonically isomorphic to $R_{\mathfrak{p}} \otimes_R A$.*

**Example 4.3.** *Let $R$ be an integral domain with field of fractions $K$. For every $R$-module $A$, we define the $K$-vector space $KA = K \cdot A := S^{-1}A$ with $S = R \setminus \{0\}$. By* [2, Proposition 2.4]*, it follows that $K \otimes_R A = KA$. In particular, we conclude that* $\operatorname{rank}(A) = \dim_K(KA)$.

**Lemma 4.4.** *If $A$ is a torsion-free $R$-module, then the canonical $R$-module homomorphism from $A$ to $KA$ is injective. In this case we identify $A$ as a submodule of $KA$.*

*Proof.* Straightforward to check. □

We defined the $K$-vector space $KA$ in *Lemma 4.3*, but it is probably not that clear why would even define this. To this end, note that vector spaces are easier to work with than modules.

**Remark 4.5.** *Note that the notation $^*$ has several meanings. If $R$ is a ring we denote the unit group of $R$ by $R^*$. If $R$ is an integral domain with field of fractions $K$ and $A$ is an $R$-module, then $A^*$ is the dual of $A$ with respect to $R$. The difference between the unit group $R^*$ and the dual of $R^*$ will be clear from context. Finally, we denote the dual of the $K$-vector space $KA$ with respect to $K$ by $(KA)^*$.*

**Lemma 4.6.** *Let $R$ be an integral domain with field of fractions $K$. If $A$ is a torsion-free $R$-module, then the map*

$$\sigma : K \cdot (A^*) \longrightarrow (KA)^*,$$
$$\frac{f}{t} \longmapsto \left(\frac{a}{s} \mapsto \frac{f(a)}{st}\right).$$

*is bijective and $K$-linear. We will therefore identify $K \cdot (A^*)$ with $(KA)^*$.*

*Proof.* It is straightforward to check that $\sigma$ is well-defined, injective and $K$-linear. We will prove that $\sigma$ is also surjective. To this end, let $f \in (K \cdot A)^*$. As $A$ is torsion-free, we identify $A$ as a submodule of $KA$ by *Lemma 4.4*, so we can consider $f\restriction_A$. Let $a_1, \cdots, a_n$ be a set of generators of $A$. By definition we have $f(a_i) \in K$ for $1 \leqslant i \leqslant n$, so there exist $s_i \in R$ and $t_i \in R \setminus \{0\}$ such that $f(a_i) = s_i/t_i$. Now define $t := \prod_{i=1}^{n} t_i$, it follows that $t \neq 0$ as $R$ is an integral domain. By construction we have $tf(a_i) = r_i/1$ for certain $r_i \in R$. As all the generators of $A$ are sent to $R$ by $t \cdot (f\restriction_A)$, we conclude that all elements of $A$ are sent to $R$, hence $t \cdot (f\restriction_A) \in A^*$. It follows that

$$\sigma\left(\frac{t \cdot (f\restriction_A)}{t}\right) = \left(\frac{a}{s} \mapsto \frac{t \cdot f(a)}{t \cdot s}\right) = \left(\frac{a}{s} \mapsto \frac{f(a)}{s}\right) = \left(\frac{a}{s} \mapsto f\left(\frac{a}{s}\right)\right) = f,$$

so $\sigma$ is indeed surjective. □

We now have enough tools for proving the main result of this section.

**Lemma 4.7.** *Every finitely generated torsion-free module over an integral domain is torsionless.*

*Proof.* Let $R$ be an integral domain with field of fractions $K$ and let $A$ be a finitely generated torsion-free $R$-module. It is straightforward to check that

$$A^* = \{f \in K \cdot (A^*) : f(A) \subseteq R\},$$

by using *Lemma 4.4*. Substituting $A$ by $A^*$ and using *Lemma 4.6* twice yields

$$\begin{aligned} A^{**} &= \{f \in K \cdot (A^{**}) : f(A^*) \subseteq R\} \\ &= \{f \in (K \cdot A)^{**} : f(A^*) \subseteq R\}, \end{aligned}$$

where we used that $A^*$ is torsion-free, since $A$ is torsion-free. Note that $K \cdot A$ is a finite-dimensional vector space as $A$ is a finitely generated module. We conclude that $K \cdot A$ is canonically isomorphic to $(K \cdot A)^{**}$ by [6, Theorem 6.8]. It follows that

$$\begin{aligned} A^{**} &= \{x \in K \cdot A : \mathrm{ev}_x(A^*) \subseteq R\} \\ &= \{x \in K \cdot A : \forall f \in A^* : f(a) \in R\}. \end{aligned}$$

Since $A$ is torsion-free, we can identify $A$ as a submodule of $KA$ by *Lemma 4.4*. Moreover, for every $a \in A$ and $f \in A^*$, we clearly have $f(a) \in R$, so $A$ is a submodule of $A^{**}$. It follows that $A$ is torsionless. □

4.1. **Continuation of concrete examples.** We will now look at a concrete example using the trace dual, but before we do this, we introduce some notation.

**Notation 4.8.** *Let $P$ be a PID and $R$ a $P$-algebra. If $R \cong_P P^2$, then there exist $q, r \in P$ such that $R \cong P[X]/(X^2 - qX - r)$ as rings, by Lemma 3.2. We denote the* ***discriminant*** *of $X^2 - qX - r$ by $D$, so $D = q^2 + 4r$.*

**Corollary 4.9.** *Let $P$ be a PID and $R$ a $P$-algebra. If $R$ is an integral domain such that $R \cong_P P^2$, then $R$ is isomorphic to $P\left[\frac{\sqrt{D}+D}{2}\right]$ as rings, with $D$ not a square.*

*Proof.* By *Lemma 3.2*, there exist $q, r \in P$ such that $R \cong P[X]/(f)$ as rings with $f = X^2 - qX - r$. If $D$ is a square, then the roots of $f$ are contained in $P$, hence $f$ can be written as $f = f_1 f_2$ for two monic polynomials $f_1, f_2 \in P[X]$. By *Remark 3.3*, it follows that $R$ is not a domain, which is a contradiction. We conclude that $D$ is not a square, hence

$$R \cong P[X]/f \cong P\left[\frac{q + \sqrt{D}}{2}\right] \cong \mathbf{Z}\left[\frac{D + \sqrt{D}}{2}\right],$$

as $q \equiv q^2 \equiv D \mod 2$. □

**Notation 4.10.** *Let $P$ be a PID with field of fractions $Q$ and let $R$ be an integral domain such that $R$ is a $P$-algebra. Choose a $\sqrt{D} \in R$. If $R \cong_P P^2$, then the field of fractions of $R$ is given by $K = Q[\sqrt{D}]$ with $D$ as in Notation 4.8. For every $\alpha \in K$ we define the multiplication-by-$\alpha$-map $M_\alpha$ as*

$$\begin{aligned} M_\alpha : K &\to K \\ \beta &\mapsto \alpha\beta. \end{aligned}$$

**Notation 4.11.** *In the situation of Notation 4.10, if $\alpha = a + b\sqrt{D} \in K$, then $M_\alpha$ is a $Q$-linear endomorphism of $K$. We define the **trace of** $\alpha$ as*

$$\operatorname{tr}(\alpha) := \operatorname{tr}(M_\alpha) = \operatorname{tr}\begin{pmatrix} a & bD \\ b & a \end{pmatrix} = 2a \in Q.$$

*Indeed, for $\beta = c + d\sqrt{D} \in K$, we have*

$$\begin{aligned} M_\alpha\left(\begin{pmatrix}1 & \sqrt{D}\end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix}\right) &= M_\alpha(\beta) = \alpha\beta = (a + b\sqrt{D})(c + d\sqrt{D}) \\ &= \begin{pmatrix}1 & \sqrt{D}\end{pmatrix}\begin{pmatrix} a & bD \\ b & a \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix}. \end{aligned}$$

*Moreover, we define the **trace dual of** $R$ as*

$$\mathcal{D}^{-1} := \{x \in K : \operatorname{tr}(xR) \subseteq P\}.$$

**Lemma 4.12.** *In the situation of Notation 4.10, if $R \cong_P P^2$, then $\mathcal{D}^{-1} = \delta^{-1}R$ with $\delta = \sqrt{D}$.*

*Proof.* We have $R = P + \omega P$, where $\omega^2 = q\omega + r$, with $q$ and $r$ as in the proof of *Lemma 3.2*. Note that in order to prove the lemma, it is sufficient to show that, given an $x \in K$, we have $\delta^{-1}x \in \mathcal{D}^{-1}$ if and only if $x \in R$. So let $x = a + b\omega \in K$ with $a, b \in Q$. As 1 and $\omega$ generate $R$ over $P$, we have $\delta^{-1}x \in \mathcal{D}^{-1}$ if and only if $\operatorname{tr}(\delta^{-1}x), \operatorname{tr}(\delta^{-1}\omega x) \in P$. Using $\sqrt{D} = -q + 2\omega$ we compute $\operatorname{tr}(\delta^{-1}x) = b$ and $\operatorname{tr}(\delta^{-1}\omega x) = a + bq$. It follows that $\delta^{-1}x \in \mathcal{D}^{-1}$ if and only if $a, b \in P$, that is, if and only if $x \in R$, and the claim follows. □

**Lemma 4.13.** *In the situation of Notation 4.10, if $R \cong_P P^2$, then $K = QR$ with $K$ the field of fractions of $R$ and $Q$ the field of fractions of $P$.*

*Proof.* First of all, we have the ring inclusions $R \subseteq QR \subseteq K$. As $K$ is the smallest field containing $R$, it suffices to prove that $QR$ is a field. Finite field extensions are algebraic by [4, Corollary 21.6], and because $K/Q$ is a finite field extension by *Notation 4.10*, we conclude that $a$ has a minimal irreducible polynomial $f$ over $Q$. By clearing out the denominators we can write $f$ as $f = \sum_{0 \leqslant i \leqslant n} a_i x^i$ with $a_i \in P$ and $a_n \neq 0$. If $a_0 = 0$, then it follows that $f(x) = a_1 x$, hence $a = 0$. Otherwise, let $b = -a_1^{-1}\sum_{0 \leqslant i \leqslant n-1} a_{n+1}a^n \in QR$. We have $ab = 1$ by construction, thus $a$ is invertible. □

**Notation 4.14.** *Let $P$ be a PID with field of fractions $Q$. If $A$ is a $P$-module, then we denote the **dual of** $A$ **with respect to** $P$ as $A^\dagger := \operatorname{Hom}_P(A, P)$. Moreover, if $QA$ is a $Q$-vector space, then we denote the **dual of** $QA$ **with respect to** $Q$ as $(QA)^\dagger := \operatorname{Hom}_Q(QA, Q)$. Note that the difference will be clear from context.*

**Lemma 4.15.** *In the situation of Notation 4.8, if $A$ is a finitely generated $R$-module, then $(KA)^*$ is isomorphic to $(QA)^\dagger$ as $Q$-vector space via the map*

$$\begin{aligned} \gamma : (KA)^* &\to (QA)^\dagger, \\ f &\mapsto (\operatorname{tr} \circ f). \end{aligned}$$

*Proof.* It is straightforward to check that $\gamma$ is injective. By *Lemma 4.13* it follows that $KA = QRA = QA$. Note that $QA$ is a finite dimensional vector space, hence $(QA)^* = (KA)^*$ and $(QA)^\dagger$ are both $Q$-vector spaces with the same dimension as $QA$. It follows that $\gamma$ is an isomorphism of $Q$-vector spaces. □

**Lemma 4.16.** *In the situation of Notation 4.10, if $A$ is a finitely generated torsion-free $R$-module, then $A^\dagger = \gamma(\delta^{-1}A^*)$, with $\gamma$ the map of Lemma 4.15 and $\delta = \sqrt{D}$, with $D$ as in Notation 4.8.*

*Proof.* By *Lemma 4.6*, we conclude $Q \cdot (A^\dagger) = (Q \cdot A)^\dagger$, and just as in *Lemma 4.7*, it follows that

$$A^\dagger = \left\{ f \in (Q \cdot A)^\dagger : f(A) \subseteq P \right\}.$$

Rewriting the equality using *Lemma 4.6*, *Lemma 4.12* and *Lemma 4.15* yields

$$\begin{aligned} A^\dagger &= \gamma(\{g \in KA^* : \forall x \in A : \operatorname{tr}(g(x)) \in P\}) \\ &= \gamma(\{g \in KA^* : \forall x \in A : \forall y \in R : \operatorname{tr}(yg(x)) \in P\}) \\ &= \gamma(\left\{g \in KA^* : \forall x \in A : g(x) \in \delta^{-1}R\right\}). \end{aligned}$$

Finally note that $g(A) \in \delta^{-1}R$ if and only if $g \in \delta^{-1}A^*$, hence the claim follows. □

**Lemma 4.17.** *In the situation of Notation 4.10, if $A$ is a finitely generated $R$-module, then $A^{\dagger\dagger} := (A^\dagger)^\dagger = A^{**}$.*

*Proof.* Just as in *Lemma 4.7*, it follows that

$$A^{\dagger\dagger} = \left\{ x \in QA : \forall f \in A^\dagger : f(x) \in P \right\}.$$

Rewriting the equation using $QA = KA$ and *Lemma 4.16* yields

$$A^{\dagger\dagger} = \left\{ x \in KA : \forall g \in A^* : \forall y \in R : \operatorname{tr}(y\delta^{-1}g(x)) \in P \right\}.$$

Finally, by *Lemma 4.12*, it follows that

$$\begin{aligned} A^{\dagger\dagger} &= \left\{ x \in KA : \forall g \in A^* : \delta^{-1}g(x) \in \delta^{-1}R \right\} \\ &= \{ x \in KA : \forall g \in A^* : g(x) \in R \}, \end{aligned}$$

where the latter set is $A^{**}$ by *Lemma 4.7.* □

**Corollary 4.18.** *Let $P$ be a PID and let $R$ be an integral domain and a $P$-algebra such that $R \cong_P P^2$. If $A$ is a finitely generated torsion-free $R$-module, then $A$ is reflexive.*

*Proof.* It follows that $A$ is a finitely generated torsion-free $P$-module. As $P$ is a PID, we can use the structure theorem of finitely generated torsion-free modules over PIDs, see [4, Theorem 16.5], to conclude that $A$ is a free $P$-module. If $V$ is a finite-dimensional vectorspace, then $V = V^{**}$, by [6, Theorem 6.8]. The proof of this theorem only uses that $V$ has a finite basis, hence the theorem also holds for finitely generated free $P$-modules. The claim then follows immediately by *Lemma 4.17.* □

If $P$ is a PID and $R$ is a domain such that $R \cong P^2$ as $P$-modules, then every ideal of $R$ can be generated by two elements by *Lemma 3.1*. By *Theorem 2.16*, it follows that every ring extension $S$ of $R$ in $K$, that is finitely generated as an $R$-module, is reflexive. We therefore conclude that the corollary proven above follows from a part of *Theorem 2.16* ((1) implies (2)), that we will not prove in this thesis.

## 5. Proof of one implication of Matlis' local theorem

We shall follow the proof of Matlis most of the time and include more details whenever we feel that it is necessary. Let $R$ be an integral domain with field of fractions $K$. Recall that condition (2) of *Theorem 2.16* states that every ring extension $S$ of $R$ in $K$ that is finitely generated as an $R$-module is a reflexive ring such that $\bigcap_{n\geqslant 0} I^n = 0$ for every $R$-ideal $I \subsetneq S$.

**Theorem 5.1.** *Let $R$ be a local Noetherian integral domain. If* (2) *holds, then $R$ has property FD, that is, every finitely generated torsion-free $R$-module is a direct sum of ideals.*

To prove the theorem, we will use the following recursively.

**Proposition 5.2.** *Let $R$ be a local Noetherian integral domain such that* (2) *holds. If $A$ is a finitely generated torsion-free $R$-module of rank greater than* 1*, then $A$ is decomposable.*

*Proof that Proposition 5.2 implies Theorem 5.1.* We will prove *Theorem 5.1* by induction on the rank of a finitely generated torsion-free module. If a module has rank 0, then clearly it is an empty direct sum of ideals. Moreover, finitely generated torsion-free modules of rank 1 are isomorphic to ideals. We use *Proposition 5.2* to conclude that finitely generated torsion-free modules of rank greater than 1 are decomposable. It then follows that a finitely generated torsion-free module of rank greater than 1 can be written as a direct sum of two non-zero finitely generated torsion-free modules. As the rank of modules is additive by *Lemma B.3* of the appendix, the rank of these two modules is lower than the rank of the module we started with. By the induction hypothesis, we conclude that these two modules can be written as direct sums of ideals, hence so can the module we started with. We conclude that any finitely generated torsion-free module is a direct sum of ideals, that is, $R$ has property FD. □

*Sketch of the proof of Proposition 5.2.* Let $A$ be a finitely generated torsion-free module of rank greater than 1. We will construct a chain of strictly increasing local rings. This chain does not have to end, but if it does, then the last ring is a principal ideal domain. Moreover, we will prove that either $A$ is a module for every ring in this chain or $A$ is decomposable. If the chain ends, then $A$ is decomposable by the structure theorem for modules over principal ideal domains. If the chain does not end, then $A$ is a module for every ring in the the chain and we take the union of all the rings. The union is not finitely generated, but it will be isomorphic to a submodule of $A$. Over Noetherian rings, submodules of finitely generated modules are finitely generated, which contradicts the fact that the union is not finitely generated. □

### 5.1. Proof that finitely generated torsion-free modules are decomposable.

We will now prove *Proposition 5.2.* To this end, let $R$ be a local Noetherian integral domain and let $A$ be a finitely generated torsion-free $R$-module with rank greater than 1. We define the trace ideal of $A$ as $I := \sum_{f\in A^*} f(A)$, and we claim that $I$ is indeed an ideal of $R$. To this end, note that $I$ is an additive subgroup of $R$ as $I$ is just a sum of subgroups of $R$. As $rf(a) = f(ra)$ for every $r \in R, a \in A$ and $f \in A^*$, it follows that $I$ is an ideal. As $R$ is a local ring with maximal ideal $M$, there are only two cases: $I = R$ and $I \subseteq M$, and we consider both cases separately.

5.1.1. *The case where the trace ideal equals the whole ring.* First assume $I = R$. By the following lemma it follows that $A$ is decomposable.

**Lemma 5.3.** *Let $R$ be a local integral domain and let $A$ be an $R$-module with rank greater than 1. Denote the trace ideal of $A$ by $I = \sum_{f \in A^*} f(A)$. If $I = R$, then $A$ is decomposable.*

*Proof.* By the definition of $I$, it follows that there exists an $n \in \mathbf{N}$ and elements $f_1, \dots, f_n \in A^*$ and $a_1, \dots, a_n \in A$ such that $1 = \sum_{i=1}^{n} f_i(a_i)$. Since $R$ is a local ring, the sum of two non-units in $R$ is a non-unit by *Lemma A.2* of the appendix. We conclude that there exists an integer $i$ with $1 \leqslant i \leqslant n$ such that $f_i(a_i)$ is an unit of $R$. It follows that there exist $a \in A$ and $f \in A^*$ such that $f(a) = 1$. Note that $f(0) = 1$ implies $R = 0$, which contradicts the assumption that $R$ is a domain, thus if $f(a) = 1$ for an $a \in A$ and $f \in A^*$, then $a \in A \setminus \{0\}$. Let these $f$ and $a \neq 0$ be given and define

$$g : R \to A,$$
$$r \mapsto ra.$$

Finally, consider the short exact sequence

$$0 \longrightarrow R \xrightarrow{g} A \xrightarrow{\text{mod } Ra} A/Ra \longrightarrow 0.$$

Since for every $r \in R$ we have $f(g(r)) = f(ra) = rf(a) = r$, we conclude that this short exact sequence splits by [4, Theorem 9.3], thus there exists an isomorphism $A \cong_R R \oplus A/Ra$. We will now show that neither $R = 0$ nor $A/Ra = 0$ holds. First of all, since $R$ is a domain, we have $R \neq 0$. Finally, as we assumed that $A$ is an $R$-module whose rank exceeds 1, we get a contradiction if $A/Ra = 0$, as this would imply that $A \cong_R Ra \cong_R R$, where we used that $ra \neq 0$ for all $r \in R \setminus \{0\}$, as $A$ is torsion-free and $a \neq 0$. We therefore conclude that neither $R = 0$ nor $A/Ra = 0$ holds, thus $A$ is decomposable. $\square$

5.1.2. *The case where the trace ideal is contained in the maximal ideal.* Now assume $I \subseteq M$. If $M$ is invertible, then $R$ is an principal ideal domain by *Lemma A.4* of the appendix. Every finitely generated $R$-module $A$ of a principal ideal domain $R$ can be decomposed as $A \cong_R T(A) \oplus R^r$, with $T(A)$ the torsion submodule of $A$ and $r$ the rank of $A$ by [4, Theorem 16.5]. It follows that $A$ is decomposable as the rank of $A$ exceeds 1. If $M$ is not invertible, then $R_1 := M^{-1}$ is a ring extension of $R$ by *Lemma A.3* of the appendix. We will now show that $A$ is an $R_1$-module.

**Lemma 5.4.** *Let $R$ be a reflexive local integral domain with field of fractions $K$ and maximal ideal $M$. Let $A$ be a finitely generated torsion-free $R$-module with trace ideal $I$, and assume $I \subseteq M$. If $M$ is not invertible, then $R_1 = M^{-1}$ is a ring extension of $R$ and $A$ is an $R_1$-module.*

*Proof.* For any $r \in R_1$, and $a \in A$ define

$$ra : A^* \to K,$$
$$f \mapsto rf(a).$$

Since $I \subseteq M$, it immediately follows that $R_1 = M^{-1} \subseteq I^{-1}$, and combining this with $f(a) \in I$ yields $rf(a) \in R$, hence $ra \in A^{**}$. Every finitely generated module of an integral domain has finite rank by *Lemma B.2* of the appendix. As $A$ is also torsion-free, it follows that $A$ is torsionless by *Lemma 4.7.* Since $R$ is reflexive, every torsionless $R$-module of finite rank is reflexive, hence $A$ is reflexive. We conclude that $A = A^{**}$, hence $ra \in A$. It follows that $A$ is an $R_1$-module, and the operation of $R_1$ on $A$ is the natural action inherited from the operation of $K$ on $KA$. $\square$

Finally consider the following proposition, which will be proven in *Section 5.2*

**Proposition 5.5.** *Let $R$ be a local Noetherian domain with field of fractions $K$. If $R$ satisfies condition (2) of Theorem 2.16, then we construct a possibly finite strictly increasing chain of subrings of $K$*

$$R \subsetneq R_1 \subsetneq R_2 \subsetneq \dots,$$

*where $R_{i+1} = M_i^{-1}$ with $M_i$ the maximal ideal of $R_i$. Each ring is local except possibly for the last one if the chain ends. If the chain ends, then the last ring is a principal ideal domain. Moreover, each ring is generated by two elements as a module over the previous one.*

All the conditions of *Proposition 5.5* are satisfied, so let the chain of rings be as given in the proposition. By the following lemma, we conclude that either $A$ is an $R_n$-module for every $n \geqslant 0$ or $A$ is decomposable.

**Lemma 5.6.** *Let $R$ be an integral domain and let $A$ an $R$-module with rank greater than 1. Assume that all the conditions of Proposition 5.5 are satisfied, and let the chain of subrings be as given in the lemma. Then either $A$ is an $R_n$-module for every $n \geqslant 0$, or $A$ is decomposable.*

*Proof.* We will prove that $A$ is an $R_n$-module for every $n \geqslant 0$, where $R_0 = R$, by induction. First of all, the case $n = 0$ holds by assumption. Let $N \in \mathbf{Z}_{\geqslant 1}$ be given and assume that $A$ is an $R_n$ module for all $n < N$. We will prove that $A$ is an$R_N$ module by showing that $R_{N-1}$ satisfies all the conditions of *Lemma 5.4*.

As $R_{N-1}$ is a subring of $K$, it follows that $R_{N-1}$ is an integral domain. Since $R_{N-1}$ is not the last ring in the chain, we conclude that it is local. Moreover, as $A$ is a finitely generated $R$-module, it immediately follows that $A$ is a finitely generated $R_{N-1}$ module, as $R_{N-1}$ is a ring extension of $R$. Since $R_{N-1}$ is a ring extension of $R$ contained in $K$, and $A$ is a torsion-free $R$-module, it immediately follows that $A$ is a torsion-free $R_{N-1}$-module. As $R_{N-1}$ is finitely generated as an $R$-module, it follows that $R_{N-1}$ is reflexive by condition (2). Furthermore, as $R$ and $R_{N-1}$ are both contained in the same field of fractions $K$, the rank of $A$ as an $R_{N-1}$-module is the same as the rank of $A$ as an $R$ module, which exceeds 1.

Let $I$ be the trace ideal of $A$ as an $R_{N-1}$-module. If $I = R_{N-1}$, then $A$ is decomposable by *Lemma 5.3*. Hence without loss of generality, we assume $I \subseteq M_{N-1}$.

If $M_{N-1}$ is invertible, then $R_{N-1}$ is an principal ideal domain by *Lemma A.4* of the appendix. It follows that $A$ is decomposable by the same argumentation as in the first paragraph of *Section 5.1.2*. Hence without loss of generality, we assume that $M_{N-1}$ is not invertible, so $M_{N-1}^{-1}$ is a ring extension of $R_{N-1}$ by *Lemma A.3* of the appendix. As we assumed that the trace ideal is contained in $M_{N-1}$, we conclude that $R_{N-1}$ satisfies all the conditions of *Lemma 5.4*, so $A$ is an $R_N$ module. By induction, we conclude that $A$ is an $R_n$ module for all $n \geqslant 0$. □

If the chain of rings ends, then $R_n$ is a principal ideal domain for some $n \in \mathbf{Z}_{\geqslant 0}$, which implies that $A$ decomposes just as in the first paragraph of *Section 5.1.2*, by [4, Theorem 16.5]. Without loss of generality we therefore assume that the chain does not end. We then define $S := \bigcup_{n \geqslant 0} R_n$, with $R_0 = R$. It follows that $S$ is an integral domain, and we will prove that $S$ is not finitely generated as an $R$-module.

We will prove this claim by contradiction, so assume there exist $s_1, ..., s_n \in S$ with $S = \sum_{i=1}^n s_i R$. For every $s_i$ there exists a smallest index $j$ such that $s_i \in R_j$, and let $m$ be the maximum of these smallest indices. We then have $s_i \in R_m$ for all $1 \leqslant i \leqslant n$, so by the definition of an $R$-module, we have $S = \sum_{i=1}^{n} s_i R \subseteq R_m$, which implies $S = R_m$. For every $k \geqslant m$, it follows that $R_k = R_m$, so the chain ends, which is a contradiction as we assumed that the chain does not end.

Note that $A$ is an $R_n$-module for every $n$, such that the operation of $R_n$ on $A$ is the natural one extending the operation of $R$ on $A$ for all $n$, hence $A$ is an $S$-module. For every $a \in A$ we then have $Sa \subseteq A$, so $Sa$ is an $R$-submodule of $A$ for every $a \in A$. As $A$ is a torsion-free $R$-module, we conclude that $A$ is a torsion-free $K$-module. Since $S$ is contained in $K$, it immediately follows that $A$ is a torsion-free $S$-module. Hence, if $a \neq 0$, then $S$ is isomorphic to $Sa$. As $R$ is Noetherian, it follows that $R$-submodules of finitely generated $R$-modules are finitely generated by [2, Proposition 1.4]. We now have a contradiction, as $Sa$ is finitely generated and isomorphic to $S$ as an $R$-module, while $S$ is not finitely generated as an $R$-module. This finishes the proof of *Proposition 5.2*. □

5.2. **Some technical results.** Before we prove *Proposition 5.5* about the existence of the chain of rings, we first prove several other results.

**Theorem 5.7** (Hilbert Basis Theorem)**.** *If a ring $R$ is Noetherian, then $R[X]$ is Noetherian.*

*Proof.* For a proof, see [2, Theorem 1.2]. □

We will use the following corollary of the Hilbert Basis Theorem extensively.

**Corollary 5.8.** *If $R_0$ is a Noetherian ring, and $R$ is a finitely generated algebra over $R_0$, then $R$ is Noetherian.*

*Proof.* For a proof, see [2, Corollary 1.3]. □

**Lemma 5.9.** *Let $R$ be an integral domain with field of fractions $K$ and let $I$ be a fractional ideal of $R$. If $f \in I^*$, then there exists an $x \in K$ such that $f = (i \mapsto xi)$.*

*Proof.* As $I$ is a fractional ideal, there exists a non-zero $r \in R$ such that $rI \subseteq R$. For any $i, j \in I$ we then have $rif(j) = f(rij) = rjf(i)$. In particular, if $i, j \neq 0$, then $x := f(i)/i = f(j)/j \in K$ and the claim follows immediately. □

**Lemma 5.10.** *Let $R$ be an integral domain. If $I$ is a non-zero fractional $R$-ideal, then Lemma 5.9 gives us a natural isomorphim $I^{-1} \cong I^*$ of $R$-modules.*

*Proof.* Consider

$$\varphi : I^{-1} \to I^*,$$
$$x \mapsto (i \mapsto xi).$$

It is straightforward to check that $\varphi$ is an injective $R$-module homomorphism. We will now show that $\varphi$ is also surjective, so let $f \in I^*$. By *Lemma 5.9*, there exists an $x \in K$ such that for all $i \in I$ we have $f(i) = xi \in R$. This $x$ satisfies $xI \subseteq R$, hence $x \in I^{-1}$. □

**Lemma 5.11.** *Let $R$ be an integral domain with field of fractions $K$. If $I$ is a non-zero fractional $R$-ideal, then $I^{-1-1} = I^{**}$.*

*Proof.* For any $x \in K$ we have $x \in I^{-1-1}$ if and only if $xy \in R$ for all $y \in I^{-1}$, that is, if and only if $f(x) \in R$ for all $f \in I^*$, where we used *Lemma 5.10*. We conclude that $x \in K$ if and only if $\mathrm{ev}_x \in I^{**}$, that is, if and only if $x \in I^{**}$, and the claim follows. □

**Corollary 5.12.** *Let $R$ be a reflexive Noetherian integral domain with field of fractions $K$. If $I$ is a non-zero fractional $R$-ideal, then $I = I^{-1-1}$.*

*Proof.* For any fractional $R$-ideal $I$, it follows that $rI \subseteq R$ is an $R$-ideal. Since $R$ is Noetherian, we conclude that $rI$ is finitely generated, hence $I$ is finitely generated. As every fractional $R$-ideal is contained in $K$, we conclude that the fractional ideals of $R$ are torsion-free as $R$-modules. By *Lemma 4.7*, it follows that every fractional ideal is torsionless, and by *Lemma B.2*, it follows that every fractional ideal has finite rank. Since $R$ is reflexive, we conclude that $I = I^{**}$ for every non-zero fractional ideal $I$, and the claim then follows by *Lemma 5.11*. □

**Lemma 5.13.** *Let $R$ be a reflexive Noetherian integral domain and let $\mathfrak{I}$ be the set of non-zero fractional ideals of $R$. Then*

$$\begin{aligned} \pi : \mathfrak{I} &\to \mathfrak{I}, \\ I &\mapsto I^{-1}, \end{aligned}$$

*is an inclusion-reversing bijection.*

*Proof.* For any $I, J \in \mathfrak{I}$ such that $I \subseteq J$ we have $\pi(J) = J^{-1} \subseteq I^{-1} = \pi(I)$, so $\pi$ is inclusion-reversing. By *Corollary 5.12*, the inverse of $\pi$ is $\pi$ itself as

$$\pi(\pi(I)) = \pi(I^{-1}) = I^{-1-1} = I,$$

for all $I \in \mathfrak{I}$, so $\pi$ is bijective. □

**Lemma 5.14.** *If $R$ is a reflexive Noetherian integral domain and $M$ is a maximal ideal of $R$, then $M^{-1}/R \cong R/M$ as $R/M$-vectorspaces.*

*Proof.* By *Lemma 5.13*, we have a one-to-one correspondence between fractional ideals $I$ such that $M \subseteq I \subseteq R$ and fractional ideals $J^{-1}$ with $R^{-1} \subseteq J^{-1} \subseteq M^{-1}$. As $R$ and $R^{-1}$ are both $R$-modules such that $RR = R$ and $RR^{-1} = R$ and the inverse of an ideal is unique by *Remark 2.19*, we have $R = R^{-1}$. As fractional ideals contained in $R$ are actual ideals of $R$, and $M$ is a maximal ideal, we conclude that there are no proper non-maximal ideals $I$ such that $M \subseteq I \subseteq R$, so by the argument above, there are no fractional ideals $J^{-1}$ such that $R^{-1} \subsetneq J^{-1} \subsetneq M^{-1}$.

It is straightforward to check that $M^{-1}/R$ is an $R/M$ vector space. Since there is a one-to-one correspondence between the set of $R$-submodules of $M^{-1}$ containg $R$ and the set of $R$-submodules of $M^{-1}/R$, we conclude that $M^{-1}/R$ has no proper non-zero $R$-submodules, as we have just shown that $M^{-1}$ has no proper $R$-submodules that strictly contain $R$. In particular, we conclude that $M^{-1}/R$ has no proper non-zero $R/M$-submodules, so $M^{-1}/R$ has no proper non-zero $R/M$-subspaces. It follows that $M^{-1}/R$ is a one-dimensional $R/M$-vectorspace, hence $M^{-1}/R$ is isomorphic to $R/M$ as $R/M$-vectorspace. □

**Corollary 5.15.** *Let $R$ be a reflexive integral domain. If $M$ is a maximal ideal of $R$, then $M^{-1}$ is generated by two elements over $R$.*

*Proof.* Consider the following exact sequence of $R$-modules

$$0 \longrightarrow R \longrightarrow M^{-1} \longrightarrow M^{-1}/R \longrightarrow 0.$$

Both $R$ and $R/M$ are generated by 1 element over $R$, and because $M^{-1}/R$ is isomorphic to $R/M$ as $R/M$-vectorspace, we conclude that $M^{-1}$ can be generated by two elements as an $R$-module. □

**Lemma 5.16.** *Let $R$ be a local integral domain with non-invertible maximal ideal $M$. Then all maximal ideals of the ring $M^{-1}$ contain $M$.*

*Proof.* Let $N$ be a maximal ideal of $M^{-1}$. As $M$ is not invertible, it follows that $M$ is an $M^{-1}$-ideal. We conclude that $M+N$ is an $M^{-1}$-ideal, hence either $M+N = N$ or $M + N = M^{-1}$ holds. If $M + N = N$, then $M$ is contained in $N$ and we are done. If $M + N = M^{-1}$, we consider the inclusion

$$R/((M + N) \cap R) \hookrightarrow M^{-1}/(M + N) = 0.$$

We have $M \subseteq R$, so $(M + N) \cap R = M + (N \cap R)$. Moreover, as $N$ is a maximal ideal of $M^{-1}$ we have $1 \notin N$, thus $N \cap R \neq R$. It follows that $N \cap R$ is contained in the maximal ideal of $R$, thus $(M + N) \cap R = M + (N \cap R) = M$. In particular, we have $R/((M + N) \cap R) = R/M \neq 0$ mapping injectively to 0, which is clearly a contradiction. □

**Lemma 5.17.** *Let $R$ be an integral domain and let $S \supseteq R$ be an $R$-algebra generated by $r$ elements as an $R$-module for an $r \in \mathbf{Z}_{\geqslant 1}$. Let $M$ be a maximal $R$-ideal, and let $\mathcal{M}$ be the set of maximal ideals of $S$ that contain $M$. Then $\#\mathcal{M} \leqslant r$.*

*Proof.* We will prove the claim by contradiction, so assume $\#\mathcal{M} > r$. We then have $M \subseteq \bigcap_{N \in \mathcal{M}} N$ by *Lemma 5.16*, so the canonical ring homomorphism

$$k = R/M \hookrightarrow S/\bigcap_{N \in \mathcal{M}} N,$$

is well-defined. Since 1 does not get mapped to 0, it is a non-zero ring homomorphism from a field to a ring, hence it is injective. Note that the right-hand side is a $k$-vectorspace. Moreover, since $S$ can be generated by $r$ elements over $k$, we conclude that the dimension of the right-hand side as a $k$-vector space is less than or equal to $r$. Suppose $\mathcal{M}' \subseteq \mathcal{M}$ has $r + 1$ elements. As distinct maximals ideals are pairwise coprime, the chinese remainder theorem yields a ring isomorphism

$$S/\bigcap_{N \in \mathcal{M}'} N \cong \prod_{N \in \mathcal{M}'} (S/N),$$

which is also a $k$-module isomorphism. Because $S/N \neq 0$ for all $N \in \mathcal{M}'$, the right hand side is a $k$-vectorspace with dimension greater than or equal to $r + 1$, which is a contradiction. □

Combining *Lemmas 5.16 and 5.17*, we get the following.

**Corollary 5.18.** *Let $R$ be a local integral domain with non-invertible maximal ideal $M$. If $A$ is an $R$-algebra generated by $n$ elements as an $R$-module, then $A$ is a semi-local ring with at most $n$ distinct maximal ideals.* □

**Corollary 5.19.** *In the situation of Lemma 5.17, if $\#\mathcal{M} = r$, then $\bigcap_{N\in\mathcal{M}} N = M$.*

*Proof.* If $\#\mathcal{M} = r$, then we have canonical $k$-module homomorphisms
$$k = R/M \hookrightarrow S/M \twoheadrightarrow S/\bigcap_{N\in\mathcal{M}} N,$$
where $S/M$ has dimension less than or equal to $r$ as a $k$-vectorspace, and the right-hand side has dimension greater than or equal to $r$ as a $k$-vectorspace. We therefore conclude that $S/M$ and the right-hand side are canonically isomorphic, hence $\bigcap_{N\in\mathcal{M}} N = M$. □

**Lemma 5.20.** *Let $R$ be a semi-local integral domain. If $I$ is a non-zero $R$-ideal, then $I$ is invertible if and only if $I$ is principal.*

*Proof.* If $I$ is principal, then there exists an $x \in I$ such that $xR = I$. If we define $J := x^{-1}R$, then $IJ = xRx^{-1}R = xx^{-1}RR = R$, hence $I$ is invertible.

If $I$ is invertible, then $II^{-1} = R$. Let $M_1, \ldots, M_n$ be the maximal ideals of $R$. For all $i$ with $1 \leqslant i \leqslant n$ there exist $a_i \in I$ and $b_i \in I^{-1}$ such that $a_i b_i \notin M_i$. By the Chinese remainder theorem, there exist $\lambda_i \in \left(\bigcap_{j\neq i} M_j\right) \setminus M_i$. Finally define $a := \sum_{1\leqslant i\leqslant n} \lambda_i a_i$ and $b := \sum_{1\leqslant j\leqslant n} \lambda_j b_j$. We claim that $ab$ is not contained in any maximal ideal.

To this end, assume there exists a $k$ with $1 \leqslant k \leqslant n$ such that $\sum_{i,j} \lambda_i\lambda_j a_i b_j \in M_k$. If $i$ and $j$ are not both equal to $k$, then $\lambda_i\lambda_j a_i b_j$ is contained in $M_k$. By subtracting all these terms from $ab$, we conclude that $\lambda_k\lambda_k a_k b_k$ is contained in $M_k$. As $M_k$ is a prime ideal, it follows that either $\lambda_k$ or $a_k b_k$ is contained in $M_k$. Both cases lead to a contradiction, thus we conclude that $ab$ is not contained in $M_k$ for any $k$ with $1 \leqslant k \leqslant n$. It follows that $ab$ is a unit, hence
$$(a) \subseteq I \subseteq abI \subseteq aII^{-1} = aR = (a),$$
and $I = (a)$. □

**Lemma 5.21.** *Let $R$ be a local Noetherian integral domain with maximal ideal $M$. Moreover, assume* (2) *of Theorem 2.16 holds. If $M$ is not invertible, then $M$ is a principal ideal of $R_1 = M^{-1}$.*

*Proof.* For every integral ideal $I$ of $R_1$, we denote its inverse as an $R_1$-ideal by
$$I^{\#} := \{x \in K : xI \subseteq R_1\}.$$
Note that $M$ is an $R_1$-ideal, as we assumed that $M$ is not invertible. We will prove the lemma by contradiction, so assume $M$ is not a principal ideal of $R_1$. By *Corollary 5.15*, we conclude that $R_1$ is finitely generated over $R$. As ring extensions that are finitely generated as modules over local rings are semi-local by *Corollary 5.18*, it follows that $R_1$ is a semi-local ring. Since $M$ is not a principal ideal of $R_1$, we conclude that $M$ is not an invertible ideal of $R_1$ by *Lemma 5.20*, hence $MM^{\#} \neq R_1$.

As $MM^{\#}$ is an $R_1$, it follows that $MM^{\#}$ is contained in a maximal ideal $N$ of $R_1$, so let such an $N$ be given. We then have
$$(N^{\#}M)M^{\#} = N^{\#}(MM^{\#}) \subseteq N^{\#}N \subseteq R_1.$$
Since $M^{\#\#} = \left\{x \in K : xM^{\#} \subseteq R_1\right\}$, we conclude $N^{\#}M \subseteq M^{\#\#}$.

As $R_1$ is a ring extension of $R$ in $K$ that is finitely generated as an $R$-module, it follows that $R_1$ is a reflexive ring by (2) of *Theorem 2.16*. As $R_1$ is Noetherian by *Corollary 5.8*, we can use *Lemma 5.12*, which implies $M^{\#\#} = M$. It follows that $N^{\#}M \subseteq M^{\#\#} = M$, so $N^{\#} \subseteq M^{-1} = R_1$. The inclusion $R_1 \subseteq N^{\#}$ holds by definition, hence we have equality: $N^{\#} = R_1$. As $N^{\#\#} = N$, it follows that

$$N = N^{\#\#} = (N^{\#})^{\#} = (R_1)^{\#} \supseteq R_1,$$

which is a contradiction as $N$ is a maximal ideal of $R_1$. □

**Lemma 5.22.** *Let $R$ be a local Noetherian integral domain with maximal non-invertible ideal $M$ and assume (2) of Theorem 2.16 holds. If $R_1 = M^{-1}$ is not a local ring with maximal ideal $N \supsetneq M$, then $R_1$ is a principal ideal domain.*

*Proof.* Assume $R_1$ is not a local ring with maximal ideal $N \supsetneq M$. As $R_1$ is generated by two elements as an $R$-module by *Corollary 5.15*, we conclude that $R_1$ contains at most two maximal ideals by *Corollary 5.18*. We will consider two cases: the ring $R_1$ is local, or $R_1$ has two distinct maximal ideals. In the first case, it follows that $R_1$ is a local ring with maximal ideal $M$, as we assumed that $R_1$ is not a local ring with maximal ideal $N \supsetneq M$.

As $M$ is not invertible as an $R$-ideal, we conclude that $R_1$ is a ring extension of $R$ by *Lemma A.3* of the appendix, hence $R_1$ is Noetherian by *Corollary 5.8*. Moreover, we conclude that $M$ is a principal ideal of $R_1$ by *Lemma 5.21*. By (2) of *Theorem 2.16*, we have $\bigcap_{n \geqslant 0} M^n = 0$, and by using the same arguments as in the last paragraph of the proof of *Lemma A.4* of the appendix, we conclude that $R_1$ is a principal ideal domain.

In the latter case, the ring $R_1$ has two maximal ideals $N_1, N_2$, and by *Lemma 5.16* it follows that $M \subseteq N_1$ and $M \subseteq N_2$. By *Corollary 5.19* we then have $M = N_1 \cap N_2$, and because distinct maximal ideals are coprime we conclude that $M = N_1 N_2$. By using *Lemma 5.20* and *Lemma 5.21*, it follows that $N_1$ and $N_2$ are principal ideals of $R_1$.

Since $R_1$ is a ring extension of $R$ in $K$ that is finitely generated as an $R$-module, it follows that $R_1$ is a reflexive ring such that $\bigcap_{n \geqslant 0} I^n = 0$ for all ideals $I \subsetneq R_1$, since we assumed (2) of *Theorem 2.16*. Let $I \subsetneq R_1$ be an ideal. It follows that there exist $n_i \in \mathbf{N}$ such that $I \subseteq N_i^{n_i}$ and $I \not\subseteq N_i^{n_i+1}$ with $i \in \{1, 2\}$.

As $N_1$ and $N_2$ are coprime, so are $N_1^{n_1}$ and $N_2^{n_2}$, hence $I \subseteq N_1^{n_1} \cap N_2^{n_2} = N_1^{n_1} N_2^{n_2}$. Multiplying both sides by $N_1^{-n_1} N_2^{-n_2}$ yields $N_1^{-n_1} N_2^{-n_2} I \subseteq R$. If $N_1^{-n_1} N_2^{-n_2} I$ is contained in $N_1$, then $I \subseteq N_1^{n_1+1} N_2^{n_2} \subseteq N_1^{n+1}$, which is a contradiction. By symmetry, we conclude that $N_1^{-n_1} N_2^{-n_2} I$ is not contained in $N_2$, hence $N_1^{-n_1} N_2^{-n_2} I = R$. We therefore conclude $I = N_1^{n_1} N_2^{n_2}$, which is a principal ideal as both $N_1$ and $N_2$ are. It follows that $R_1$ is a principal ideal domain. □

We remind the reader that the goal of this section was to prove *Proposition 5.5* about the existence of the chain of rings used in *Section 5.1.2*. We finally have enough tools to prove it, and for convenience we state the proposition again.

**Proposition 5.23** (Proposition 5.5)**.** *Let $R$ be a local Noetherian domain with field of fractions $K$. If $R$ satisfies condition (2) of Theorem 2.16, then we construct a possibly finite strictly increasing chain of subrings of $K$*

$$R \subsetneq R_1 \subsetneq R_2 \subsetneq \dots,$$

*where $R_{i+1} = M_i^{-1}$ with $M_i$ the maximal ideal of $R_i$. Each ring is local except possibly for the last one if the chain ends. If the chain ends, then the last ring is a principal ideal domain. Moreover, each ring is generated by two elements as a module over the previous one.*

*Proof.* We construct the chain recursively. Denote $R$ by $R_0$, and let $M_i$ be the maximal ideal of $R_i$. Let $n \in \mathbf{Z}_{\geqslant 0}$ and assume we constructed the chain as in the proposition up to and including $R_n$.

First of all note that $R_n$ is a ring extension that is finitely generated as an $R$-module, as each ring is generated by two elements as module over the previous one. If $M_n$ is invertible, then $R_n$ is a principal ideal domain by *Lemma A.4*. Hence, without loss of generality, we assume that $M_n$ is not invertible, which implies that $R_{n+1} = M_n^{-1}$ is a ring extension of $R_n$ by *Lemma A.3*. Note that $R_{n+1}$ is a subring of $K$, hence $R_{n+1}$ is an integral domain. In particular, we conclude that $R_{n+1}$ is an $R$-module. As $R_n$ is reflexive, it follows that $R_{n+1}$ is generated by two elements over $R_n$ as an $R$-module by *Corollary 5.15*. By *Corollary 5.8* it then follows that $R_{n+1}$ is Noetherian.

If $R_{n+1}$ is a local ring, then either $M_n$ is the maximal ideal of $R_{n+1}$, or $M_n$ is strictly contained in the maximal ideal of $R_{n+1}$, as proven in *Lemma 5.16*. Note that if $M_n$ is the maximal ideal of $R_{n+1}$, then the chain stops. By *Lemma 5.22* it follows that if $R_{n+1}$ is not a local ring, then $R_{n+1}$ is a principal ideal domain. Finally note that every ring extension $R_{n+1} \subseteq S \subseteq K$ that is finitely generated as an $R_{n+1}$-module is also finitely generated as an $R$-module. The claim then follows by induction. $\square$

## 6. Proof of the implication of *Section 5* in a global setting.

In the section, we will give the first part of a proof of *Theorem 5.1* in a more global setting. Recall that condition (2) of *Theorem 2.16* states that every ring extension $S$ of $R$ in $K$ that is finitely generated as an $R$-module is a reflexive ring and satisfies $\bigcap_{n\geqslant 0} I^n = 0$ for every $R$-ideal $I \subsetneq S$.

**Theorem 6.1.** *Let $R$ be a Noetherian integral domain with Krull dimension at most 1. Moreover, assume that the integral closure $R'$ of $R$, inside its field of fractions $K$, is a finitely generated $R$-module. If (2) holds, then $R$ has property FD.*

This is a slightly simplified version of one of the implications proven in [1]. Bass does not require $R$ to be an integral domain, but uses the weaker assumption that $R$ has no nilpotent elements instead. To prove this theorem, the following will be used recursively.

**Proposition 6.2.** *Let $R$ be a reflexive Noetherian integral domain with Krull dimension at most 1. Moreover, assume that the integral closure $R'$ of $R$, inside its field of fractions $K$, is a finitely generated $R$-module. If $A$ is a reflexive finitely generated $R$-module of rank greater than 1, then either $A$ is decomposable, or $A$ is an $S$-module for a ring extension $S$ of $R$ that satisfies $R \subsetneq S \subseteq R'$.*

Before we prove a part of the proposition, we first need some other results.

**Lemma 6.3.** *Let $R$ be an integral domain. If $A$ is a non-zero reflexive $R$-module with trace ideal $U := \sum_{f\in A^*} f(A)$, then $U^{-1}$ is a ring extension of $R$ and $A$ is a $U^{-1}$-module.*

*Proof.* Let $a = \sum f_i(a_i) \in U$ and $x \in U^{-1}$. We have $xf_i(A) \subseteq R$ for all $i$, hence $xf_i \in A^*$ for all $i$. It follows that $xa = \sum (xf_i)(a_i) \in U$, thus $U^{-1}U \subseteq U$. We then have

$$U^{-1}U^{-1}U \subseteq U^{-1}U \subseteq U,$$

so $U^{-1}$ is a ring extension of $R$ and $U$ is a $U^{-1}$-module. Moreover, it also follows that $A^* = \mathrm{Hom}_R(A,R) = \mathrm{Hom}_R(A,U)$ is a $U^{-1}$-module. Finally note that, if $f \in \mathrm{Hom}(A^*,R) = A^{**}$ and $x \in U^{-1}$, then

$$xf(A^*) = f(xA^*) \subseteq f(A^*) \subseteq R,$$

hence $xf \in A^{**}$. As $A$ is reflexive, we conclude $xf \in A$, hence $A$ is a $U^{-1}$-module. □

**Lemma 6.4.** *Let $R$ be a Noetherian domain with Krull dimension at most 1 with field of fractions $K$. Let $R'$ be the integral closure of $R$ inside $K$. If $R'$ is a finitely generated $R$-module, then there are only finitely many maximal ideals $m \subsetneq R$ such that $R_m \neq R'_m$.*

*Proof.* Let $s \neq 0$ be the product of the denominators of the generators, it follows that $sR \subseteq sR' \subseteq R$. If $sR_m = R_m$ then $sR_m = sR'_m$ hence $R_m = R'_m$, so it suffices to prove that the set $\{m : sR_m \neq R_m\}$ is finite. As $sR_m \neq R_m$ if and only if $s \notin R_m^*$, that is, if and only if $s \in m$, we conclude

$$\{m : sR_m \neq R_m\} = \{m : s \in m\} = \{\text{maximal ideals of}\, R/(s)\}\,.$$

Every chain of prime ideals of $R/(s)$ correspondents to a chain of prime ideals of $R$ containing $s$. It follows that $\dim(R/(s)) \leqslant \dim(R) - 1$, as we can always add the prime ideal (0) to a chain of prime ideals of $R/(s)$. We conclude $\dim(R/(s)) \leqslant 0$, hence every prime ideal of $R/(s)$ is maximal. By [2, Theorem 2.14], it follows that $R/(s)$ has only finitely many maximal ideals. □

**Lemma 6.5.** *In the situation of Lemma 6.3, if there are only finitely many maximal ideals $m \subsetneq R$ with the property that $R_m \neq R'_m$, then there exist $f \in A^*$ and $x \in A$ such that $f(x) \notin m$ for all maximal ideals $m \subsetneq R$ with $R_m \neq R'_m$ and $U \not\subseteq m$.*

*Proof.* Let $m_1, \dots, m_n$ be all the maximal $R$-ideals that satisfy both $R_{m_i} \neq R'_{m_i}$ and $U \not\subseteq m_i$ for $1 \leqslant i \leqslant n$. For all $i$, there exist $f_i \in A^*$ and $x_i \in A$ such that $f_i(x_i) \notin m_i$. By the Chinese remainder theorem, there exist $y_i \in \bigcap_{j \neq i} m_j \setminus m_i$. Now define $f = \sum_{1 \leqslant i \leqslant n} y_i f_i$ and $x = \sum_{1 \leqslant i \leqslant n} y_i x_i$. By the same argumentation as in the last paragraph of the proof of *Lemma 5.20*, replacing $\lambda_i$ by $y_i$ and $f_i(x_j)$ by $a_i b_j$, it follows that $f(x)$ is not contained in $M_i$ for all $i$. □

**Lemma 6.6.** *Let $R$ be an integral domain. Then the short exact sequence*

$$0 \longrightarrow A \longrightarrow B \xrightarrow{g} C \longrightarrow 0$$

*of $R$-modules splits if and only if it splits locally at every maximal ideal of $R$.*

*Proof.* Define

$$I = \{r \in R : \exists s : C \to B : g \circ s = r \cdot \mathrm{id}_C\}.$$

It is straightforward to check that $I$ is an $R$-ideal, and the short exact sequence splits if and only if $1 \in I$, that is, if and only if $I = R$. It follows that the short exact sequence splits if and only there exist no maximal $R$-ideal $m$ such that $I \subseteq m$, that is, if and only if there exist no maximal $R$-ideal $m$ such that $I_m \subseteq mR_m$, if and only if $I_m = R_m$ and $1 \in I_m$ for every $R$-ideal $m$. The claim then follows immediately. □

Due to time constraints, we will only sketch the proof of *Proposition 6.2.*

*Sketch of the proof of Proposition 6.2.* If $U$ is the trace ideal of $A$, then $U^{-1}$ is a ring extension of $R$ and $A$ is a $U^{-1}$-module by *Lemma 6.3.* As $U^{-1}$ is a fractional ideal, there exists a non-zero $r \in R$ such that $rU^{-1} \subseteq R$ is an $R$-ideal. Since $R$ is Noetherian, we conclude that $rU^{-1}$ is finitely generated as an $R$-module, hence so is $U^{-1}$. Finite ring extensions are integral, so $U^{-1}$ is a finitely generated $R$-module that satisfies $R \subseteq U^{-1} \subseteq R'$.

If $U^{-1} \neq R$, then we're done, hence assume $U^{-1} = R$. As $U$ is a non-zero $R$-ideal, it follows that $U = U^{-1^{-1}}$ by *Lemma 5.12*, hence $U = U^{-1^{-1}} = R^{-1} = R$.

By *Lemma 6.4*, there are only finitely many maximal ideals $m \subsetneq R$ that satisfy $R_m \neq R'_m$. By *Lemma 6.5*, there exist $f \in A^*$ and $x \in A$ such that $f(x) \notin m$ for all maximal ideals $m \subsetneq R$ with $R_m \neq R'_m$. As $R \setminus R^*_m = m$ for all maximal ideals $m \subsetneq R$, it follows that $f(x) \in R^*_m \cap R$ for all maximal ideals $m \subsetneq R$ satisfying $R_m \neq R'_m$. We then claim that the short exact sequence

$$0 \longrightarrow (Kx \cap A) \xrightarrow{i} A \xrightarrow{g} A/(Kx \cap A) \longrightarrow 0,$$

splits locally for all maximal ideals $m \subsetneq R$ with $R_m \neq R'_m$. To this end, one defines a section $s$ from $A_m$ to $(Kx \cap A)_m$ using $f$ and $x$. To show that the short exact sequence also splits for maximal ideal $m$ with $R_m = R'_m$, one checks that $R_m$ is a PID by checking the conditions of *Proposition 5.5.* By *Lemma 6.6*, the short exact sequence splits globally, hence $A = (Kx \cap A) \oplus (A/(Kx \cap A))$. Since neither $(Kx \cap A) = A$ nor $(Kx \cap A) = 0$ holds, we conclude that $A$ is decomposable. □

## Appendix A. Local rings

**Definition A.1.** *Let $R$ be a commutative ring. A* ***maximal ideal*** *$M \subseteq R$ is a non-zero $R$-ideal such that the only $R$-ideal strictly containing $M$ is $R$. A* ***local ring*** *is a commutative ring with exactly one maximal ideal, and a* ***semi-local ring*** *is a commutative ring with finitely many maximal ideals.*

**Lemma A.2.** *If $R$ is a local ring with maximal ideal $M$, then $M = R \setminus R^*$.*

*Proof.* If $x \in R \setminus R^*$, then $(x) \subsetneq R$, hence $x \in (x) \subseteq M$. If $x \in M$, then $(x) \subseteq M$. As $1 \notin M$, it follows that $x \in R \setminus R^*$. □

**Lemma A.3.** *Let $R$ be a local domain with maximal ideal $M$. If $M$ is not invertible, then $R_1 := M^{-1}$ is a ring extension of $R$.*

*Proof.* First of all, we have $R \subseteq R_1$ by the definition of $R_1$. Moreover, as $M^{-1}$ is an $R$-submodule of $K$, the addition on $R_1$ is an extension of the addition on $R$. Finally note that if $I$ and $J$ are $R$-modules, then so are $I^{-1}$ and $IJ$. We conclude that $M^{-1}M$ is an $R$-module contained in $R$, hence $M^{-1}M$ is an $R$-ideal. As $M$ is not invertible, we have $M^{-1}M \neq R$, thus $M^{-1}M$ is contained in the maximal ideal $M$. Since $M$ is contained in $M^{-1}M$, as $1 \in M^{-1}$, we have equality, hence

$$M^{-1}M^{-1}M = M^{-1}M = M \subseteq R,$$

which implies that $M^{-1}M^{-1}$ is contained in $M^{-1}$. We conclude that multiplication on $R_1$ yields elements of $R_1$. □

**Lemma A.4.** *Let $R$ be a local integral domain such that $\bigcap_{n \geqslant 0} I^n = 0$ for all ideals $I \subsetneq R$. Let $M$ be the maximal ideal of $R$. If $M$ is invertible, then $R$ is a principal ideal domain.*

*Proof.* Assume $M$ is invertible and let $x \in M \setminus M^2$. Note that such an $x$ exists, as $\bigcap_{n \geqslant 0} M^n = 0$ and $M \neq 0$. We have $xM^{-1} \subseteq R$, and because $xM^{-1}$ is an $R$-module, we conclude that $xM^{-1}$ is an $R$-ideal.

As we assumed that $R$ is a local ring, it follows that either $xM^{-1} \subseteq M$ or $xM^{-1} = R$ holds. In the first case, we can multiply both sides by $M$ to get $x \in M^2$, which is a contradiction as we assumed $x \in M \setminus M^2$. We therefore conclude that $xM^{-1} = R$, and again multiplying both sides by $M$, we get $xR = xM^{-1}M = RM = M$, hence $M = (x)$.

We have $\bigcap_n M^n = 0$, so for every non-zero $R$-ideal $J$, there exists an $n \in \mathbf{N}$ such that $J \subseteq M^n$ and $J \not\subseteq M^{n+1}$. Multiplying both sides by $M^{-n}$ we get $JM^{-n} \subseteq R$ and $JM^{-n} \not\subseteq M$, and since $JM^{-n}$ is an ideal of $R$, we conclude $JM^{-n} = R$. Multiplying both sides by $M^n$ yields $J = RM^n = M^n = (x)^n = (x^n)$. □

## Appendix B. Finitely generated modules and finite rank

**Definition B.1.** *Let $R$ be an integral domain and let $A$ be an $R$-module. We say that $A$ is* ***finitely generated*** *if there exist $a_1, \ldots, a_n \in A$ such that for all $a \in A$, there exist $r_1, \ldots, r_n \in R$ with $a = \sum_{i=1}^{n} r_i a_i$.*

**Lemma B.2.** *Let $R$ be an integral domain and let $A$ be an $R$-module. If $A$ is finitely generated then $A$ has finite rank.*

*Proof.* First assume $A$ is finitely generated and let $a_1, \ldots, a_n \in A$ be a generating set for $A$. It follows that every $a$ can be written as $a = \sum_{i=1}^{n} r_i a_i$ for certain $r_i \in R$. We now claim that $\{(a_i \otimes 1) : 1 \leqslant i \leqslant n\}$ is a generating set for $A \otimes_R K$. To this end, let $a \otimes k \in A \otimes_R K$, rewriting then yields

$$a \otimes k = \left( \sum_{i=1}^{n} r_i a_i \right) \otimes k = \sum_{i=1}^{n} (r_i a_i \otimes k) = \sum_{i=1}^{n} r_i \left( a_i \otimes k \right) = \sum_{i=1}^{n} r_i k \left( a_i \otimes 1 \right),$$

hence $\operatorname{rank}(A) \leqslant n$, so $A$ has finite rank. □

**Lemma B.3.** *Let $R$ be an integral domain and let*

$$0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0,$$

*be a short exact sequence of $R$-modules. Then* $\operatorname{rank}(A) = \operatorname{rank}(A') + \operatorname{rank}(A'')$.

*Proof.* For any multiplicatively closed subset $S$ of $R$, the localization of $S$ at $R$ denoted by $S^{-1}R$ is flat as an $R$-module by [2, Proposition 2.5], so tensoring with $K$ preserves exact sequences. In particular, we conclude that

$$0 \longrightarrow K \otimes_R A' \longrightarrow K \otimes_R A \longrightarrow K \otimes_R A'' \longrightarrow 0,$$

is a short exact sequence of $K$-vector spaces. By the additivity of the dimension of vector spaces, it follows that

$$\dim_K(K \otimes A) = \dim_K(K \otimes A') + \dim_K(K \otimes A''),$$

and the claim follows. □

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