**Motivation**

**Definition 1** The modular polynomial of prime level \( p \) is a polynomial

\[
\Phi_p(X,Y) \in \mathbb{Z}[X,Y]
\]

which, for all \( \tau \in \mathbb{H} \), satisfies

\[
\Phi_p(j(\tau), j(p\tau)) = 0,
\]

where \( j(\tau) \) is the \( j \)-invariant for elliptic curves.

- Given the \( j \)-invariant \( j \) of an elliptic curve over \( \mathbb{Q} \), we can find the \( j \)-invariants of all those elliptic curves which are \( p \)-isogenous to it by computing the roots of \( \Phi_p(X,Y) \in \mathbb{K}(Y) \).
- Analogues have been computed so that given an \( Igusa \) invariant of a genus 2 curve, we can find the Igusa invariants of all those genus 2 curves which are \( p \)-isogenous to it; these analogues have huge coefficients and are difficult to handle in practice.
- In our work, we add the constraint of real multiplication, and compute modular polynomials for genus 2 which are much smaller and easier to handle. We also give a theoretical algorithm to compute modular polynomials for abelian varieties of any dimension.

**An application - isogeny graphs**

- Using the structure of isogeny graphs together with our modular polynomials we have a fast method for computing endomorphism rings.
- The isogeny graphs that we have for abelian varieties without taking into account the real multiplication do not in general have a nice structure.
- Taking into account real multiplication gives a ‘nice’ structure in many (maybe all) cases, here is an example computed by Sorina Ionica using Magma:

**Example in genus 2**

**Setup**

- Let \( F \) be a totally real number field of degree \( g \) over \( \mathbb{Q} \).
- Let \( \mathcal{O}_F \) be the maximal order of \( F \), and let \( \mathcal{O}_F^\ast \) be its trace dual.
- Let \( \mathbb{H}^g \) denote \( g \) copies of the complex upper half plane.
- Let \( SL(\mathcal{O}_F \oplus \mathcal{O}_F^\ast)^2 \) be the matrix group given by

\[
\left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in SL(\mathcal{O}_F) : a, d \in \mathcal{O}_F, b \in \mathcal{O}_F^\ast, c \in (\mathcal{O}_F^\ast)^{-1}.
\]

**Definition 2** Let \( A \) be abelian variety of genus \( g \), with a principal polarization given by \( \xi : A \to A^t \) and real multiplication via the embedding \( e : \mathcal{O}_F \to \End(A) \) such that the image of \( \mathcal{O}_F \) in \( \End(A) \) is stable under the Rosati involution. Then we say that \((A, \xi, i)\) is a principally polarized abelian variety of genus \( g \) with real multiplication by \( \mathcal{O}_F \).

We can define an action of \( SL(\mathcal{O}_F \oplus \mathcal{O}_F^\ast)^2 \) on \( \mathbb{H}^g \) under which the moduli space of principally polarized complex abelian varieties of genus \( g \) with real multiplication by \( \mathcal{O}_F \) is given by

\[ SL(\mathcal{O}_F \oplus \mathcal{O}_F^\ast)^2 \mathbb{H}^g. \]

- Denote by \( M_p \) the \( \mathbb{C} \)-algebra of Hilbert modular forms for \( SL(\mathcal{O}_F \oplus \mathcal{O}_F^\ast)^2 \).
- Denote by \( Q(M_p) \) the \( \mathbb{C} \)-algebra of quotients of elements of \( M_p \) of equal weight (inside the fraction field).

**Definition 3** Let \((A, \xi, i)\) be a principally polarized complex abelian variety of genus \( g \) with real multiplication by \( \mathcal{O}_F \) which corresponds to \( \tau \in SL(\mathcal{O}_F \oplus \mathcal{O}_F^\ast)^2 \mathbb{H}^g \) under the moduli correspondence. Fix an \( r \)-tuple \((J_1, \ldots, J_r) \in (Q(M_p))^g \) such that, for every \( \tau \in SL(\mathcal{O}_F \oplus \mathcal{O}_F^\ast)^2 \mathbb{H}^g \), the \( r \)-tuple \((J_1(\tau), \ldots, J_r(\tau)) \in (\mathbb{C}(1))^g \) determines \((A, \xi, i)\) up to isomorphism. We will call

\[ (J_1(\tau), \ldots, J_r(\tau)) \]

the isomorphism invariant of \((A, \xi, i)\). This is our analogue of the \( j \)-invariant for elliptic curves.

**The algorithm**

**INPUT:**

- An integer \( g \geq 2 \), and a totally real number field \( F \) of degree \( g \) over \( \mathbb{Q} \).
- An appropriate choice of functions \( \{J_1, \ldots, J_r\} \) for \( Q(M_p) \), and \( q \)-expansions for each numerator and denominator.
- A totally positive prime element \( \mu \) of \( \mathcal{O}_F \).

**OUTPUT:**

- A set of \( r^2 \) polynomials

\[
\begin{align*}
G_1(X_1, X_2, Z_1, Z_2) &

H_{1,2} \in \mathbb{Q}(X_1, X_2, Z_1, Z_2).
\end{align*}
\]

where the \( H_{i,j} \) are linear in \( Z_1, Z_2, Z_3, \ldots, Z_{g} \).

- This looks like the modular polynomial for elliptic curves: after carefully defining a \( \mu \)-isogenous in an analogous way to a \( p \)-isogeny for elliptic curves, but taking into account the real multiplication and the polarizations, we can deduce the following:

For \((A, \xi, i)\) a principally polarized abelian variety with isomorphism invariant \((J_1(\tau), \ldots, J_r(\tau))\), define

\[
S := \{ (J_1(\mu \tau), \ldots, J_r(\mu \tau), Z_1, Z_2) \in \mathbb{Q}(X_1, X_2, Z_1, Z_2) \mid \mu \in \mathbb{Q}(1) \}
\]

Then generally

\[
(A, \xi, i) \leftrightarrow (A', \xi', i') \quad \text{is} \ \mu \text{-isogenous to} \quad (A,\xi, i)
\]

is its isomorphism invariant \((J_1(\mu \tau), \ldots, J_r(\mu \tau))\) is a common zero of the polynomials in \( S \).

**References**