Constructing genus 2 curves over finite fields with a prescribed number of points

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Joint work with Marco Streng

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Reminder: Elliptic Curves over finite fields

Definition

Let E be an elliptic curve over a finite field $\mathbb{F}_q$. The *q-power Frobenius morphism* on E is defined to be

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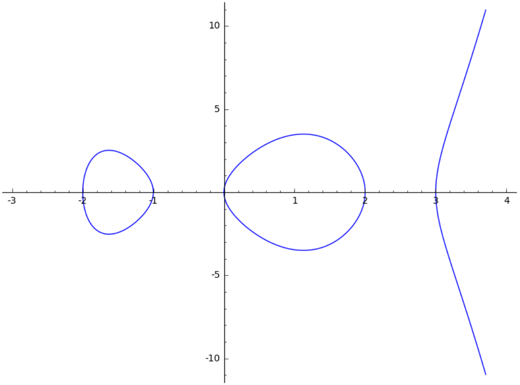
- The roots of $H_K(X)$ mod q are the j-invariants of all the elliptic curves $E/\mathbb{F}_q$ such that $\mathrm{End}(E) = \mathcal{O}_K$.
- There is an algorithm to enumerate all the elliptic curves with $1 - t + q$ points given those with endomorphism ring $\mathcal{O}_K$.

Genus 2 curves

- A genus 2 curve C over a finite field $\mathbb{F}_q$, with q odd, has a hyperelliptic model

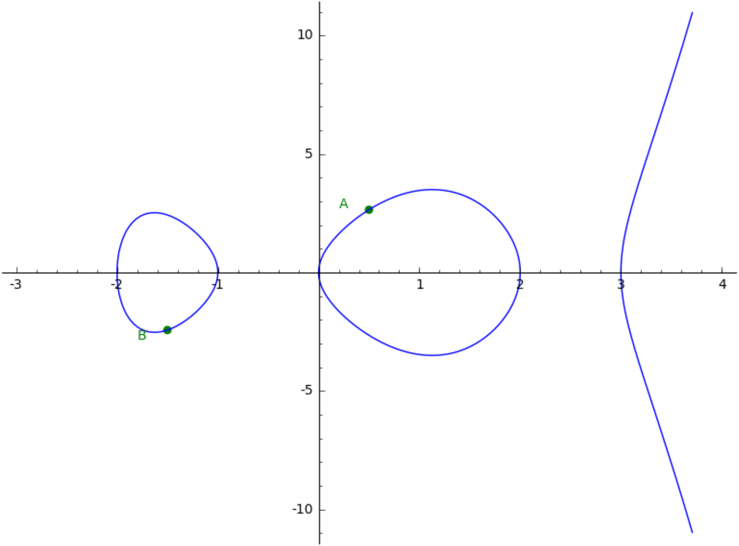
$$y^2 = f(x) \in \mathbb{F}_q[x],$$

where $\deg(f) = 5$ or 6.



The group law for genus 2 curves

We define a group law on genus 2 curves with *pairs of points*.



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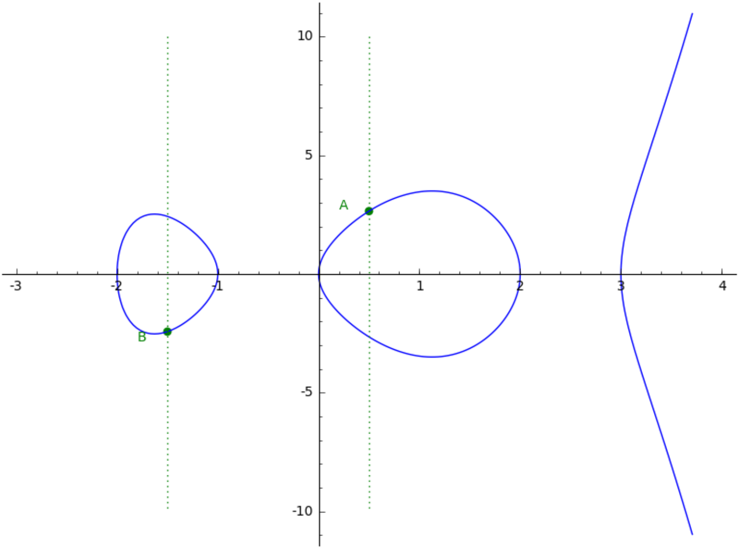

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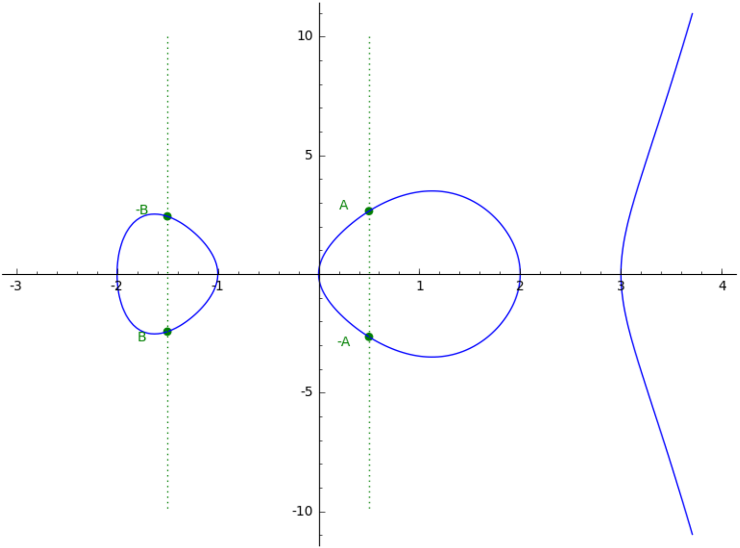

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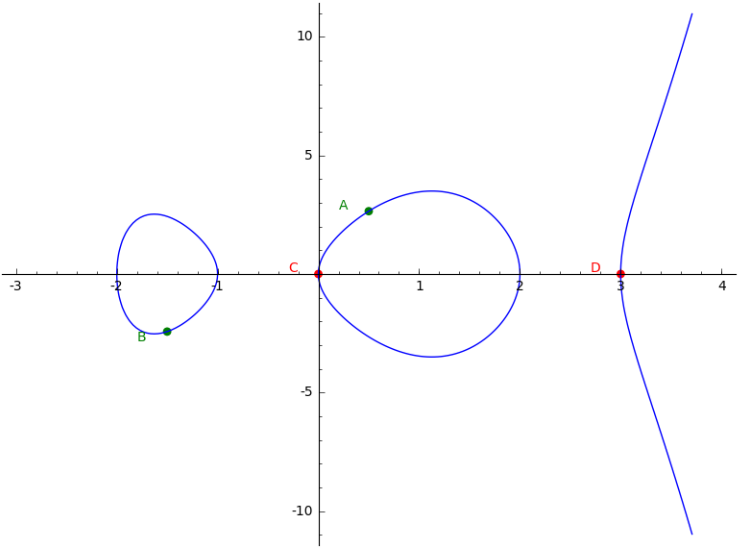
Suppose we have another pair of points $\{C, D\}$:



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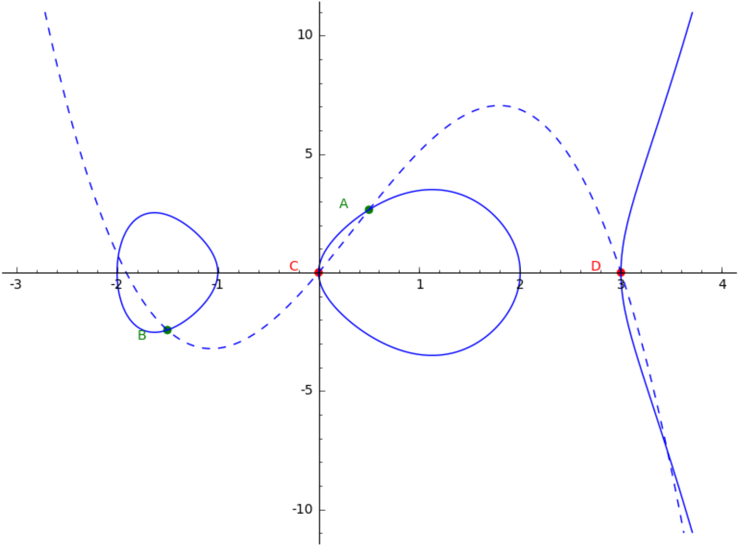
Draw the unique cubic passing through A, B, C, D:



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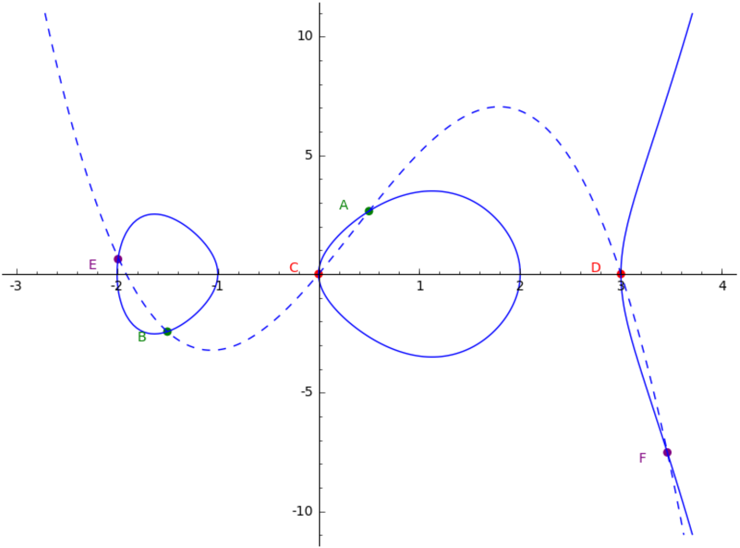
We define $\{A, B\} + \{C, D\} + \{E, F\} = 0$.



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- $\#\mathrm{Jac}(C)(\mathbb{F}_q) = 1 - t + 2q + s - tq + q^2$

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Counting points on genus 2 curves over finite fields

Suppose that π is a complex (non-real) root of $\chi_{\pi_q}(X) = X^4 - tX^3 + (2q+s)X^2 - tqX + q^2$. Then

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Constructing elliptic curves with a given number of points

Strategy: construct elliptic curves $E/\mathbb{F}_q$ such that $\mathrm{End}(E) = \mathcal{O}_K$.

Definition

The *class polynomial* for K is defined to be

$$H_K(X) = \prod_{E/\mathbb{C}:\mathrm{End}(E)=\mathcal{O}_K} (X - j(E)) \in \mathbb{Z}[X].$$

- This polynomial has integral coefficients!
- The roots of $H_K(X)$ mod q are the j-invariants of all the elliptic curves $E/\mathbb{F}_q$ such that $\mathrm{End}(E) = \mathcal{O}_K$.
- There is an algorithm to find *all* the elliptic curves with $1 - t + q$ points given all the elliptic curves with endomorphism ring $\mathcal{O}_K$.

Interlude: Invariants of genus 2 curves

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- We give an algorithm to construct many more genus 2 curves with $1 - t + q$ points given all the genus 2 curves with endomorphism ring $\mathcal{O}_K$.

Our contributions

- We give a new algorithm to compute the class polynomials for genus 2 curves that mimics the current state-of-the-art for genus 1.

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- I hope to use class polynomials to construct (families of) pairing-friendly genus 2 curves.

Thank you!